THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-fifth regular meeting of the Society was held in New York City on Saturday, February 27, 1915. The attendance at the two sessions included the following thirty-nine members:

Professor Joseph Bowden, Professor E. W. Brown, Dr. T. H. Brown, Professor B. H. Camp, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor Elizabeth B. Cowley, Dr. Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. H. Garrett, Professor O. E. Glenn, Professor C. C. Grove, Professor G. H. Hallett, Professor H. E. Hawkes, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Mr. P. H. Linehan, Professor James Maclay, Dr. H. F. MacNeish, Dr. E. J. Miles, Mr. G. W. Mullins, Dr. G. A. Pfeiffer, Dr. H. W. Reddick, Professor R. G. D. Richardson, Mr. P. R. Rider, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Edwin R. Smith, Professor Oswald Veblen, Mr. R. A. Wetzel, Miss E. C. Williams.

The President of the Society, Professor Ernest W. Brown, occupied the chair, being relieved at the afternoon session by Vice-President Oswald Veblen. The Council announced the election of the following persons to membership in the Society: Professor J. V. Balch, Bethany College; Professor E. J. Berg, Union College; Mr. Millar Brainard, Chicago, Ill.; Mr. L. C. Cox, Purdue University; Mr. C. H. Forsyth, University of Michigan; Dr. H. C. Gossard, University of Oklahoma; Mr. M. S. Knebelman, Lehigh University; Dr. W. V. Lovitt, Purdue University; Dr. L. C. Mathewson, Dartmouth College; Mr. A. L. Miller, University of Michigan; Dr. Bessie I. Miller, Johns Hopkins University; Mr. I. R. Pounder, University of Toronto; Mr. L. L. Steimley, Indiana University; Mr. Chid-Cheow Yen, Tangshan Engineering College. Three applications for membership in the Society were received.

From the early days of the Society the informal dinners held in connection with the meetings have been a most valuable

supplement to the scientific programme, furnishing opportunities for a general exchange of views, conferences on questions of mathematics and of general policies, and renewing old and making new acquaintance. To promote this social side of the Society's activities, the Council has recently appointed a committee, consisting of Professors Fiske, Hawkes, and Kasner, to arrange a dinner for each New York meeting and by giving publicity to these occasions to induce a large attendance. Notice of the dinner is sent out with the programme of the meeting, and members expecting to attend are requested to fill out the accompanying card and return it to the Secretary, in order that adequate accommodations may be provided. The result at the recent annual meeting was the large attendance of seventy members. The February meeting is always, for several reasons, a smaller affair. Yet on this occasion nineteen members spent a very enjoyable evening together.

The following papers were read at this meeting:

(1) Professor M. FRÉCHET: "Sur les fonctionnelles bilinéaires."

(2) Professor A. S. HATHAWAY: "Gamma coefficients."

(3) Mr. P. H. LINEHAN: "Equilong invariants of irregular and regular analytic curves."

(4) Professor B. H. CAMP: "Multiple integrals over infinite fields."

(5) Mr. A. R. SCHWEITZER: "On the methods of mathematical discovery."

(6) Mr. P. R. RIDER: "An extension of Bliss's form of the problem of the calculus of variations, with applications to the generalization of angle."

(7) Professor E. B. WILSON: "The Ziwet-Field note on plane kinematics."

(8) Professor O. E. GLENN: "Ternary transvectant systems."

(9) Dr. E. J. MILES: "Note on the application of the calculus of variations to a problem in mechanics."

(10) Professor A. B. FRIZELL: "The permutations of the natural numbers cannot be well ordered."

(11) Mr. C. H. FORSYTH: "Osculatory interpolation formulas."

(12) Mr. J. F. RITT: "A function of a real variable with any desired derivatives at a point."

(13) Mr. J. F. RITT: "On Babbage's functional equation."

Professor Fréchet's paper was communicated to the Society through Professor D. R. Curtiss. Mr. Ritt was introduced by Professor Kasner. In the absence of the authors the papers of Professor Fréchet, Professor Hathaway, Mr. Schweitzer, Professor Wilson, Professor Frizell, and Mr. Forsyth were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Fréchet investigates operations $U_{f,\phi}$ which are linear with respect to f and to ϕ separately. Generalizing a known theorem of F. Riesz, he proves that $U_{f,\phi}$ may be represented as follows:

$$U_{f,\phi} = \int_a^{a'} \int_b^{b'} f(s)\phi(t)d_s d_t u(s, t),$$

the double integral on the right being defined as the limit of

$$\Sigma_i \Sigma_j f(\xi_i)\phi(\eta_j)\Delta_{i,j}u,$$

where

$$\Delta_{i,j}u = u(s_i, t_j) - u(s_i, t_{j-1}) - u(s_{i-1}, t_j) + u(s_{i-1}, t_{j-1}).$$

A striking result is that the function $u(s, t)$ need fulfill no other condition than that $|\Sigma_i \Sigma_j \epsilon_i \epsilon_j' \Delta_{i,j}u|$ have a finite upper bound whatever may be the signs of ϵ_i, ϵ_j', provided $|\epsilon_i| = |\epsilon_j'| = 1$. This representation by generalized double integrals is then applied to second differentials of continuous operations.

2. Professor Hathaway defines a gamma coefficient of coordinates $x, y, \cdots$ and parameters $a, b, \cdots$. The coefficient of two dimensions is

$$[axby] = (ax + by)\Gamma(x + y)/\Gamma(x + 1)\Gamma(y + 1),$$

and similarly for any dimension of coordinates.

This coefficient is shown to be constant upon an axis and, at any point, equal to the sum of its values at the points which precede it by a unit in the direction of each axis.

Taking integral values, none negative, for coordinates and parameters, an integral function is formed of corresponding arguments $p, q, \cdots$, consisting of the sum of all terms with

coefficients of a given weight $n = ax + by + \cdots$, the coordinates of a coefficient being the exponents of the corresponding arguments. If m be the greatest parameter, it is shown that this function equals the sum of the nth powers of the roots of the equation

$$x^m = px^{m-a} + qx^{m-b} + \cdots.$$

This is a generalization of the formula for s_n given by Waring and proved by Serret (Cours d'Algèbre supérieure, volume I, page 445, article 196) from the formula of Lagrange. Waring's formula is deduced from the above by taking a, b, $\cdots$, m, as the natural numbers 1, 2, $\cdots$, m. When some parameters are equal, the general formula obtained above cannot be deduced from that of Waring. For example, the above result becomes the multinomial theorem for $a = b = \cdots = m = 1$; but Waring's formula, in the same case, is $(p + q + \cdots)^n = (p + q + \cdots)^n$.

Further properties of these coefficients will be the subject of a future paper.

3. Irregular analytic curves

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots$$

(u, v being Hessian line coordinates) are shown by Mr. Linehan to possess invariants under the group of equilong transformations of the plane except when $p = 2$. For each of the three cases which arise when invariants exist, the simplest invariant is derived.

The simplest invariant of a regular analytic curve under the linear equilong transformations of the plane is also obtained.

4. Professor Camp's immediate object in this paper is to prepare a foundation for a discussion, to be given later, of multiple and iterated integrals containing parameters, in which the integrations are extended over infinite fields. For this purpose it is necessary to coordinate the various definitions of these multiple integrals which have been given in recent years, to consider more fully than has been done heretofore the conditions of their existence, their relation to the iterated integrals, and certain of their fundamental properties.

5. In a previous article* Mr. Schweitzer stated a heuristic

* *Revue de Métaphysique et de Morale*, March, 1914.

principle of comparison based on identity, or principle of unification of terms. In the present paper is stated a principle of comparison antithetic to the preceding and based on diversity. The latter is called the principle of "furcation" of terms and is stated as follows:

The existence of dissimilarities between given terms implies the existence of dissimilar general terms which underlie or embrace the particular terms.

It is important to recognize the interdependence of the two principles. Thus by replacing "dissimilarities," "dissimilar" by "similarities" (resemblances), "similar" respectively, one obtains a possible interpretation of the author's principle of unification. The principle of "furcation" is useful in providing, directly or indirectly, problems of conflict for solution by means of the principle of unification.

An interesting special case of the principle of furcation is provided by equivalent mathematical theories or concepts which diverge under generalization. An instance of this kind is the author's "trifurcative" generalization* of the betweenness relation in the foundations of geometry. Many other illustrations might be given.

A concluding part of the author's paper is devoted to a critique of a remarkable article by J. T. Merz, History of European Thought of the Nineteenth Century, volume II, pages 627–740: "Development of mathematical thought." As a critical basis, the author's article on the working hypotheses of mathematics (cited above) is used.

6. Bliss has developed a theory of the calculus of variations for integrals of the form $I = \int f(x, y, \tau)ds$, where $\tan \tau$ is the slope of the curve considered. Mr. Rider's paper extends the theory to the integral $I = \int f(x, y, z, \tau, \sigma)ds$, where σ is the angle that a space curve makes with its own projection in the xy-plane, and τ is the angle that this projection makes with the x-axis. Necessary and sufficient conditions are studied.

A transversal surface is used in giving certain generalized definitions of angle and of solid angle. Incidentally, the differential equations of geodesics and generalized geodesics on this surface are derived. The definitions of angle reduce

* *American Journal*, vol. 31 (1909), p. 366.

under proper conditions to the definitions used in euclidean geometry.

7. Referring to the adoption by Ziwet and Field of the operators i and $e^{\phi i}$ of Burali-Forti and Marcolongo, Professor Wilson points out (1) that these operators are not algebraic as their form indicates, (2) that the operator $e^{\phi u \wedge}$, which is patently non-algebraic, will not only serve the same purposes, but in addition lead to the derivation by A. C. Lunn of the Euler-Rodrigues form for rotations, and (3) that the algebraic operator $e^{\phi i}$, used in connection with the Gibbs bivector, will also accomplish the ends desired by Ziwet and Field. This note will appear in the *American Mathematical Monthly*.

8. The methods for the construction of fundamental systems of invariant formations of ternary quantics, known up to the present time, are tentative. The purpose of Professor Glenn's paper is, first, to deduce the simplest possible algorithm for the construction of such systems. Instead of deriving all invariants by means of one transvectant operation, four such operations are introduced, which may, however, be united under one major process of making cogredient substitutions in a double mixed polar. The main theorem of the paper is: If two systems of forms are finite and complete, the system derived therefrom by the above major transvectant process is finite and complete. A method of deriving the irreducible members of such a system is given. By the theorems of the paper the twenty invariant formations of two conics, for instance, may be readily written down.

9. The problem considered by Dr. Miles is: Given a chord of definite length and variable density; to find its form for uniform horizontal distribution of the mass if the center of gravity lies as low as posssible.

The density is assumed to be a function of x, say $r(x)$, and a problem in variations of the Lagrangian type results. The functions y and r are then determined and the curve is found to be a parabola—the solution given in text books on mechanics.

10. By methods developed in previous papers read before the Society (Lincoln, November, 1914; Chicago, December,

1914, and New York, January, 1915), Professor Frizell shows that the assumption that the set of permutations of the natural numbers can be well ordered leads, by a line of reasoning due to Cantor, to two sets of permutations which are both in one-to-one correspondence with the whole set of permutations of the natural numbers but cannot be put into one-to-one correspondence with each other.

11. As is well known, whenever successive intervals are to be interpolated, separate curves are used in the several intervals. In osculatory interpolation these separate curves are required to have the same slopes and curvatures at their points of intersection for purposes of smoother graduation. The appropriate formulas to be used in interpolating several values in each interval depend upon the fundamental interpolation curve upon which they are based, and formulas have been derived based upon Newton's and Everett's interpolation formulas. Mr. Forsyth derives the formulas based upon Stirling's and Bessel's formulas together with a series of corrections of fifth differences to be used in connection with the four formulas in lieu of the formulas in succeeding intervals. The paper will appear in the *Quarterly Publications* of the American Statistical Association.

12. There exists no analytic function having the sequence of numbers $a_1, a_2, \cdots, a_n$, as derivatives for $x = 0$, unless $|a_n/n!|^{1/n}$ stays bounded as n increases indefinitely.

Professor Kasner suggested the desirability of learning whether a non-analytic function can be found when no analytic function exists. Mr. Ritt has shown that the function

$$\sum_{n=1}^{n=\infty} \frac{a_n x^n}{n!}\left(1 - e^{-1/(b_n x^2)}\right),$$

where $1 < b_n > |a_n|$, has the desired sequence of derivatives.

13. About 1815, Charles Babbage attempted to find all functions of which the nth iterative is the independent variable itself. He observed that if $f(x)$ is such a function, $\varphi^{-1}f\varphi(x)$ will also be one, where $\varphi(x)$ is arbitrary; but he had no means of knowing how general his solution was.

Mr. Ritt has commenced a rigorous discussion of Babbage's functions, limiting himself to the case of the real variable,

and making a set of five assumptions. It appears that the most general solution, when n is greater than 2, is $\varphi^{-1}f^r\varphi(x)$, where the integer r is prime to n. The case $n = 2$ is discussed separately and a simple algorism is given for reducing all differentiable functions of order 2 to a single type.

F. N. COLE,
Secretary.

THE LEGENDRE CONDITION FOR A MINIMUM OF A DOUBLE INTEGRAL WITH AN ISOPERIMETRIC CONDITION.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, February 28, 1914.)

THE Legendre, or second necessary, condition for a minimum of a double integral, where there is no isoperimetric condition, has been derived by Kobb,* where the equations of the surfaces involved are in parametric form, and by Mason,† where x and y are the independent variables. The analogous condition for the isoperimetric problem has been proved to be sufficient to insure a permanent sign to the second variation,‡ but it has not been proved to be necessary.

In the present paper this condition,

$$h_{pp}(x, y, z, p, q; \lambda)h_{qq}(x, y, z, p, q; \lambda) - h_{pq}^2(x, y, z, p, q; \lambda) \geqq 0,$$

or expressed in parametric form,

$$H_{11}(x, y, z, x_u, x_v, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, x_v, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, x_v, \cdots, z_v; \lambda) \geqq 0,$$

is proved to be necessary for either a maximum or a minimum.

Given two functions $f(x, y, z, p, q)$ and $g(x, y, z, p, q)$ and a surface

$$S: \qquad z = z(x, y)$$

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* "Sur les maxima et les minima des intégrales doubles," *Acta Mathematica*, vol. 16 (1892), p. 108.

† "A necessary condition for an extremum of a double integral," BULLETIN, vol. 13 (1907), p. 293.

‡ Kobb, *Acta Mathematica*, vol. 17 (1893), p. 331.

which satisfies the Lagrange differential equation

$$(1) \qquad h_z(x, y, z, p, q; \lambda) - \frac{\partial}{\partial x} h_p(x, y, z, p, q; \lambda) - \frac{\partial}{\partial y} h_q(x, y, z, p, q; \lambda) = 0;^*$$

it is desired to find a second condition which must be satisfied if the surface S gives a value to the integral

$$(2) \qquad J = \int\int_\Omega f(x, y, z, p, q)dxdy$$

as small as that given by any other admissible surface in the neighborhood of S. The region Ω is assumed to be bounded by a curve L of class D',† without double points, and the number of its intersections with any line parallel to either of the axes is assumed to be less than a fixed constant. The function $z(x, y)$ is assumed to be of class C'' in Ω, as are also the functions f and g in the neighborhood of S. A surface is said to be admissible if it is of class D', intersects S along a space curve which projects into L, and gives the same value as S to the double integral

$$K = \int\int_\Omega g(x, y, z, p, q)dxdy.$$

It will also be assumed that $z(x, y)$ is not a solution of the equation

$$(3) \qquad g_z(x, y, z, p, q) - \frac{\partial}{\partial x} g_p(x, y, z, p, q) - \frac{\partial}{\partial y} g_q(x, y, z, p, q) = 0.$$

If a one parameter family of admissible surfaces

$$\overline{S}: \qquad z = z(x, y) + \epsilon\overline{\zeta}(x, y, \epsilon)$$

is given, where the function $\overline{\zeta}(x, y, 0)$ and its partial derivative $\zeta_\epsilon(x, y, 0)$ are of class D', and this value substituted for z in the function f, the first variation of J must vanish because of

* Bolza: Vorlesungen über Variationsrechnung, p. 662.
† Bolza, loc. cit., p. 63.

equation (1). The second variation is found to be

$$\delta^2 J = \epsilon^2 \int\int_\Omega (f_{zz}\bar{\zeta}^2 + 2f_{zp}\bar{\zeta}\bar{\zeta}_x + 2f_{zq}\bar{\zeta}\bar{\zeta}_y + f_{pp}\bar{\zeta}_x^2 + 2f_{pq}\bar{\zeta}_x\bar{\zeta}_y + f_{qq}\bar{\zeta}_y^2 + f_z\bar{\zeta}_\epsilon + f_p\bar{\zeta}_{x\epsilon} + f_q\bar{\zeta}_{y\epsilon})dxdy. \tag{4}$$

Since the surfaces considered are admissible, $\delta^2 K$, which can be evaluated in the same way, vanishes. If it is multipled by λ and added to $\delta^2 J$, and then Green's theorem* is applied, equation (4) becomes

$$\delta^2 J = \epsilon^2 \int\int_\Omega (h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + \cdots + h_{qq}\bar{\zeta}_y^2)dxdy + \epsilon^2 \int\int_\Omega (h_z - \frac{\partial}{\partial x}h_p - \frac{\partial}{\partial y}h_q)\bar{\zeta}_\epsilon\, dxdy,$$

where as usual $h = f + \lambda g$. The last integral vanishes on account of equation (1), leaving

$$\delta^2 J = \epsilon^2 \int\int_\Omega (h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + \cdots + h_{qq}\bar{\zeta}_y^2)dxdy. \tag{5}$$

It will now be proved that if there is a point on S where the inequality

$$h_{pp}h_{qq} - h_{pq}^2 < 0 \tag{6}$$

is satisfied, the function $\bar{\zeta}(x, y, \epsilon)$ can be chosen in such a way that $\delta^2 J$ will be negative, and consequently there is no minimum.

If there is such a point there must be a region including the point where inequality (6) is satisfied. Two distinct points P_0 and P_0' in such a region can then be chosen, whose coordinates will be called $x_0, y_0, z(x_0, y_0)$ and $x_0', y_0', z(x_0', y_0')$. Since $z(x, y)$ is not a solution of equation (3), it can be assumed that

$$g_z(x_0', y_0', \cdots) - \frac{\partial}{\partial x_0'}g_p(x_0', y_0', \cdots) - \frac{\partial}{\partial y_0'}g_q(x_0', y_0', \cdots) \neq 0. \tag{7}$$

* Bolza, loc. cit., p. 654.

The expression

$$h_{pp}(x_0, y_0, \cdots) \cos^2 \alpha + 2h_{pq}(x_0, y_0, \cdots) \cos \alpha \sin \alpha + h_{qq}(x_0, y_0, \cdots) \sin^2 \alpha$$

must vanish for two positive values of α less than π, and changes its sign when and only when α passes through one of them. It follows that a positive constant k^2 and a finite interval can be chosen in such a way that if α_1 and α_2 are any two angles in the interval, the inequalities

$$(8) \quad h_{pp} \cos^2 \alpha_i + 2h_{pq} \cos \alpha_i \sin \alpha_i + h_{qq} \sin^2 \alpha_i < -k^2 \qquad (i = 1, 2),$$

are satisfied at P_0, and since h_{pp}, h_{pq} and h_{qq} are continuous, they are satisfied at every point of S in a neighborhood of P_0. If there is given any positive constant δ it is possible to select two distinct angles α_1' and α_2' near to a root of the equation

$$h_{pp}(x_0', y_0', \cdots) \cos^2 \alpha + 2h_{pq}(x_0', y_0', \cdots) \cos \alpha \sin \alpha + h_{qq}(x_0', y_0', \cdots) \sin^2 \alpha = 0,$$

such that the inequalities

$$(9) \quad | h_{pp} \cos^2 \alpha_i' + 2h_{pq} \cos \alpha_i' \sin \alpha_i' + h_{qq} \sin^2 \alpha_i' | < \delta \qquad (i = 1, 2),$$

are satisfied at every point of S in a neighborhood of P_0'. For convenience these angles will be chosen so that $\alpha_2 - \alpha_1 = \alpha_2' - \alpha_1'$.

Two rhombuses R and R' will be defined as follows:* R is bounded by the lines

$$d - u_1 = 0, \quad d - u_2 = 0, \quad d + u_1 = 0, \quad d + u_2 = 0,$$

where

$$u_i = (x - x_0) \cos \alpha_i + (y - y_0) \sin \alpha_i, \quad (i = 1, 2),$$

and R' by the lines

$$d - u_1' = 0, \quad d - u_2' = 0, \quad d + u_1' = 0, \quad d + u_2' = 0,$$

where

$$u_i' = (x - x_0') \cos \alpha_i' + (y - y_0') \sin \alpha_i'.$$

* Compare with Mason, loc. cit., p. 295.

The constant d will be taken so small that R and R' do not overlap, and are entirely within the projections on the x, y-plane of the respective neighborhoods in which the inequalities (8) and (9) are satisfied. A function $\zeta(x, y)$ will be defined as identically zero outside of R, and inside of R as equal to the distance of the point x, y from the nearest side of R. That is,

$$\zeta(x, y) = d \mp u_i, \qquad (i = 1, 2),$$

where the sign and subscript of u are chosen so as to make ζ as small as possible. The function $\zeta'(x, y)$ will be chosen in the analogous way. Since $\alpha_2 - \alpha_1 = \alpha_2' - \alpha_1'$ the rhombuses R and R' are of the same size and shape and the equation

$$\int\int_R \zeta(x, y)dxdy = \int\int_{R'} \zeta'(x, y)dxdy$$

is satisfied.

The function $\bar{\zeta}(x, y, \epsilon)$ will now be defined by the equation

$$\epsilon\bar{\zeta}(x, y, \epsilon) = \epsilon\zeta(x, y) + \epsilon'(\epsilon)\zeta'(x, y),$$

where $\epsilon'(\epsilon)$ is to be determined by the condition that the surfaces $\bar{S}$ be admissible. The first variation of K is found to be equal to

$$\int\int_R \left(g_z - \frac{\partial}{\partial x}g_p - \frac{\partial}{\partial y}g_q\right)\zeta dxdy + \frac{d\epsilon'(0)}{d\epsilon}\int\int_{R'} \left(g_z - \frac{\partial}{\partial x}g_p - \frac{\partial}{\partial y}g_q\right)\zeta' dxdy = 0.$$

If the mean value theorem is applied to these integrals and the equation is solved for $d\epsilon'(0)/d\epsilon$, it becomes

$$\frac{d\epsilon'(0)}{d\epsilon} = -\frac{\left(g_z - \frac{\partial}{\partial x}g_p - \frac{\partial}{\partial y}g_q\right)_{x=\xi, y=\eta}}{\left(g_z - \frac{\partial}{\partial x}g_p - \frac{\partial}{\partial y}g_q\right)_{x=\xi', y=\eta'}},$$

where ξ, η is a point in R and ξ', η' a point in R'. The denominator cannot vanish if d is sufficiently small, because of inequality (7). Consequently a finite constant m can be chosen such that

$$\left|\frac{d\epsilon'(0)}{d\epsilon}\right| < m.$$

If the second variation of K is equated to zero, the coefficient of $d^2\epsilon'(0)/d\epsilon$ is also equal to

$$\int\int_{R'}\left(g_z - \frac{\partial}{\partial x}g_p - \frac{\partial}{\partial y}g_q\right)\zeta' dxdy \neq 0.$$

Consequently this second derivative exists. Therefore the functions

$$\bar{\zeta}(x, y, 0) = \zeta(x, y) + \frac{d\epsilon'(0)}{d\epsilon}\zeta'(x, y),$$

$$\zeta_*(x, y, 0) = \frac{d^2\epsilon'(0)}{d\epsilon^2}\zeta'(x, y)$$

exist and they have the required continuity.

When $\bar{\zeta}(x, y, \epsilon)$ is determined in this way equation (5) may be written

$$\delta^2 J = \epsilon^2\int\int_{R+R'}(h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + 2h_{zq}\bar{\zeta}\bar{\zeta}_y)dxdy$$

$$+ \epsilon^2\int\int_R(h_{pp}\cos^2\alpha_i + 2h_{pq}\cos\alpha_i\sin\alpha_i + h_{qq}\sin^2\alpha_i)dxdy$$

$$+ \epsilon^2\left(\frac{d\epsilon'(0)}{d\epsilon}\right)^2\int\int_{R'}(h_{pp}\cos^2\alpha_i' + 2h_{pq}\cos\alpha_i'\sin\alpha_i'$$

$$+ h_{qq}\sin^2\alpha_i')dxdy,$$

where i takes the values 1 and 2 in the appropriate parts of R and R'. It can now be easily proved that

$$\delta^2 J < -\epsilon^2 A(k^2 - Md(d + 4)(1 + m^2) - \delta m^2),$$

where M is the largest of the maxima of the numerical values of h_{zz}, h_{zp} and h_{zq}, and A is the area of R.* Since d and δ can be taken as small as is desired without affecting k^2, they can be taken so small that $\delta^2 J$ will be negative and there is no minimum.

In a similar way it can be proved that there is no maximum if inequality (6) is satisfied, and the following theorem will be proved:

A necessary condition that the surface S furnish either a maximum or a minimum for the double integral (2), *relative to*

* Compare with Mason, loc. cit., pp. 295–6.

the admissible surfaces, is that

$$h_{pp}(x, y, z, p, q; \lambda)h_{qq}(x, y, z, p, q; \lambda) - h_{pq}^2(x, y, z, p, q; \lambda) \geqq 0, \tag{10}$$

at every point of S.

If the surface S is represented by the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

the functions f and g must be replaced by $F(x, y\ z, x_u, x_v, \cdots z_v)$ and $G(x, y, z, x_u, x_v, \cdots, z_v)$ respectively. Then if the equations of $\bar{S}$ are written

$$x = x(u, v) + \epsilon\xi(u, v, \epsilon), \quad y = y(u, v) + \epsilon\eta(u, v, \epsilon),$$
$$z = z(u, v) + \epsilon\zeta(u, v, \epsilon),$$

the second variation becomes

$$\delta^2 J = \epsilon^2 \int\int_\Omega (H_{xx}\xi^2 + 2H_{xy}\xi\eta + \cdots + H_{z_v z_v}\zeta_v^2)dudv. \tag{11}$$

The values of J and K are assumed to be unchanged by any change in the parametric representation of S which leaves the surface S itself invariant. This furnishes a number of relations between the partial derivatives of F and G, among which are the following:*

$$F_{x_u x_u} = F_{11}X^2, \quad F_{x_u y_u} = F_{12}XY, \quad F_{x_u x_v} = F_{12}X^2,$$
$$F_{x_u y_v} + F_{x_v y_u} = 2F_{12}XY, \quad F_{x_v x_v} = F_{22}X^2, \quad F_{x_v y_v} = F_{22}XY, \tag{12}$$

and the other formulas derived from these by permuting the letters x, y, z and X, Y, Z in the same way. The functions F_{11}, F_{12} and F_{22} are continuous and X, Y and Z are the direction cosines of the normal to S.

It will now be assumed that there is a function $\omega(u, v, \epsilon)$, such that

$$\xi(u, v, \epsilon) = \omega(u, v, \epsilon)X,$$
$$\eta(u, v, \epsilon) = \omega(u, v, \epsilon)Y,$$
$$\zeta(u, v, \epsilon) = \omega(u, v, \epsilon)Z.$$

If these values are substituted in the integrand of equation (11), it becomes a quadratic form in ω, ω_u, ω_v. The coefficient

* Kneser: Lehrbuch der Variationsrechnung, p. 282.

of ω_u^2 is seen to be

$$H_{x_u x_u} X^2 + 2H_{x_u y_u} XY + H_{y_u y_u} Y^2 + 2H_{x_u z_u} XZ + 2H_{y_u z_u} YZ + H_{z_u z_u} Z^2.$$

Equations (12), with F replaced by H, reduce this to the form

$$H_{11}(X^2 + Y^2 + Z^2)^2 = H_{11}.$$

Similarly the coefficients of $\omega_u\omega_v$ and ω_v^2 can be proved equal to $2H_{12}$ and H_{22} respectively. The other coefficients will be called H_{00}, $2H_{01}$ and $2H_{02}$ respectively, and equation (11) becomes

$$\delta^2 J = \epsilon^2 \int\int_\Omega (H_{00}\omega^2 + 2H_{01}\omega\omega_u + 2H_{02}\omega\omega_v + H_{11}\omega_u^2 + 2H_{12}\omega_u\omega_v + H_{22}\omega_v^2)\,du\,dv.$$

This equation is in the same form as equation (5), and from this point on the argument is so nearly the same as in the non-parametric case that it need not be repeated here. The analogue of inequality (10) is seen to be

$$H_{11}(x, y, z, x_u, \cdots, z_v; \lambda) H_{22}(x, y, z, x_u, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, \cdots, z_v; \lambda) \geqq 0.$$

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NOTE ON THE DERIVATIVE AND THE VARIATION OF A FUNCTION DEPENDING ON ALL THE VALUES OF ANOTHER FUNCTION.

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1. IN a recent article,* Fréchet has given a treatment of the differential of a function depending on a curve, by making use of and evaluating Riesz's expression of a linear relation in terms of a Stieltjes integral. According to Fréchet, if $F_a^b[\varphi]$ depends on all the values of $\varphi(x)$ between a and b, then

* M. Fréchet, "Sur la notion de différentielle d'une fonction de ligne," *Transactions of the American Mathematical Society*, vol. 15 (1914), pp. 135–161.

if the differential of F exists it is given by the formula

$$dF[\varphi(x)] = \int_a^b \Delta\varphi(\xi)d\alpha_\phi(\xi), \tag{1}$$

where $\Delta\varphi(\xi)$ is the increment in φ, and $\alpha_\phi(\xi)$ is some function of finite variation. On the other hand, Volterra* has shown that, under certain conditions, the variation of F is given by the formula

$$\delta F[\varphi(x)] = \int_a^b F'[\varphi(x)|\xi]\delta\varphi(\xi)d\xi, \tag{2}$$

where $F'\varphi[(x) \mid \xi]$ is the functional derivative of F with respect to $\varphi(x)$ at the point ξ. The integral in (1), however, according to Fréchet, itself splits up into three parts, of which one has a form similar to (2), so that as a special case we should have

$$dF[\varphi(x)] = \int_a^b A[\varphi(x) \mid \xi]\Delta\varphi(\xi)d\xi. \tag{3}$$

It is the object of this short paper, in the first place, to derive the formula (2) under slightly less restrictive conditions than those of Volterra, and in the second place, by adopting a point of view more akin to that of Fréchet, to show the relation between equation (1) and equations (2) and (3).

2. We shall consider as a region for the argument $\varphi(x)$ that included between two given continuous functions $\Phi_1(x)$ and $\Phi_2(x)$, where $\Phi_1(x) < \Phi_2(x)$, in the interval $a \leqq x \leqq b$; i. e., the region

$$\Phi_1(x) \leqq \varphi(x) \leqq \Phi_2(x), \qquad a \leqq x \leqq b,$$

and we shall assume that $F[\varphi]$ is defined for every continuous function in that region, and is continuous.† This we shall call the assumption (α).

In addition to (α), in order to obtain formula (2), Volterra makes four assumptions I–IV. By a different method of proof, however,—the one which we first adopt—it is possible to arrive at (2) by means of (α), (II) and (III) alone, from which

* V. Volterra, "Sopra le funzioni che dipendono da altre funzioni," *Rendiconti della R. Accademia dei Lincei*, vol. 3 (1887), pp. 97–105, 141–146, 153–158.

† We mean that $F[\varphi]$ has continuity of the zeroth order.

IV follows, but not (I) in its entirety.* These assumptions are as follows.

(II) Take an interval h within ab and give to $\varphi(x)$ within h a continuous variation $\theta(x)$, of one sign, such that $|\theta(x)| < \epsilon$. Denote by ΔF the corresponding change of F, and write $\sigma = \int_h \theta(x)dx$. It is assumed that the ratio $\Delta F/\sigma$ approaches a fixed limit (denoted by $F'[\varphi(x) \mid \xi]$ and spoken of as the functional derivative of F) as ϵ and h approach zero in an arbitrary manner, provided that the interval h always contains within itself the value $x = \xi$.†

(III) It is assumed that the ratio $\Delta F/\sigma$ approaches its limit uniformly with respect to all possible functions $\varphi(x)$ and values ξ.

Let us for convenience denote by (III_1) that part of (III) which requires uniformity with respect to the functional argument alone.

I. First Deduction of Formula (2).

3. The following theorem is introductory.

Theorem 1. *If for a certain continuous function $\varphi_0(x)$ the function $F[\varphi(x)]$, continuous in $\varphi(x)$, has a derivative $F'[\varphi(x) \mid \xi]$ for every value of ξ in a certain closed sub-interval $a'b'$ of ab,‡ that derivative remains finite and continuous throughout $a'b'$.*

To prove this theorem, let $\xi_1, \xi_2, \cdots$ be any infinite set of values of ξ in $a'b'$ having ξ_0 as a limiting point, and suppose that

$$F'[\varphi_0(x) \mid \xi_0] = t, \qquad \lim_{n=\infty} F'[\varphi_0(x) \mid \xi_n] = t'.$$

We shall assume that t' is finite; the case where t' is infinite occasions an obvious modification of the proof. Let $|t-t'| = p$, and suppose momentarily $p \neq 0$.

Give to $\varphi_0(x)$ a variation of one sign, $\theta_1(x)$, of the kind specified in (II), and take ϵ_1 and h_1 so small that we have the

* See § 4, and footnote.

† We understand here that $\theta(\xi) \neq 0$. This restriction turns out to be immaterial, but makes the definition correspond more closely to that of the ordinary derivative, where a restriction somewhat related to this is essential to the nature of the operation.

This definition requires obvious modification when $\xi = a$ or b and when $\varphi(x) = \Phi_1(x)$ or $\Phi_2(x)$.

‡ In particular, we may take $a' = a$ and $b' = b$.

inequality

$$(4) \qquad |F[\varphi_0(x)+\theta_1(x)]-F[(\varphi_0(x)]-t\sigma_1| < \left|\frac{1}{2}p\sigma_1\right|.$$

We may, however, by taking a variation θ_2, small enough, and about a point ξ_n near enough to ξ_0, and adding to it a variation θ_3, small enough everywhere, yet different from zero at ξ_0, obtain a variation θ_1 for which is satisfied the inequality

$$(5) \qquad \begin{aligned} &|F[\varphi_0(x)+\theta_2(x)+\theta_3(x)]-F[\varphi_0(x)] \\ &\qquad -t'(\sigma_2+\sigma_3)| < \left|\frac{1}{2}p(\sigma_2+\sigma_3)\right|. \end{aligned}$$

For, since F is assumed to be continuous, the increment θ_3 may be made so small as to affect the difference $F[\varphi_0+\theta_2]-F[\varphi_0]$ by as little as we please. But from (4) and (5), by taking $\theta_1=\theta_2+\theta_3$, it follows that

$$|(t-t')(\sigma_2+\sigma_3)| < p|\sigma_2+\sigma_3|,$$

which is a contradiction. Hence $p=0$, and the theorem is proved.

4. Let us now make the assumptions (α), (II), (III_1). It follows at once that *the derivative* $F'[\varphi(x)\,|\,\xi]$ *is continuous in regard to* $\varphi(x)$, *if* ξ *is any fixed value in the interval* ab. For we have

$$\left|F'[\varphi(x)\,|\,\xi]-\frac{F[\varphi(x)+\theta(x)]-F[\varphi(x)]}{\sigma}\right| < \eta,$$

and on account of the condition of uniformity,

$$\begin{aligned} &|F'[\varphi_2(x)\,|\,\xi]-F'[\varphi_1(x)\,|\,\xi]| \\ &\qquad \leqq |1/\sigma\{F[\varphi_2(x)+\theta(x)]-F[\varphi_1(x)+\theta(x)]\}|+2\eta. \end{aligned}$$

Hence first fixing the ϵ and h corresponding to $\theta(x)$ small enough so that 2η is less, say, than $\omega/2$, we can then take φ_2 near enough to φ_1, the $\theta(x)$ being fixed, so that the other part of the expression is also less than $\omega/2$. That is to say, by taking $|\varphi_2(x)-\varphi_1(x)|$ small enough, we can make the left hand member of the inequality as small as we please. Hence F' has continuity of the zeroth order with respect to φ.

From this it follows that F' is continuous uniformly with respect to φ, if φ is restricted to any family of curves whose ordinates are uniformly continuous functions of a finite

number of parameters, over a perfect domain for those parameters, or if φ is restricted to a family of curves closed in the sense that the limiting curves are uniform limits; but not that F' is continuous in φ uniformly with respect to all continuous functions φ in the given region, nor even that F' is continuous in ξ uniformly for all continuous functions φ in the given region. *If however we make the whole of assumption* (III), it follows, as is easily verifiable, that *F' is continuous in ξ uniformly for all points ξ and all continuous curves φ in the region*, and hence that *F' is continuous in ξ and φ uniformly with respect to ξ and φ, provided that φ is restricted to a family of continuous functions, closed in the sense that the limiting functions are uniform limits.**

5. We can now prove, by means of the assumptions (α), (II), (III_1), something analogous to Rolle's theorem.

THEOREM 2. *Let $F[\varphi(x)]$ be a function for which* (α), (II), (III_1) *hold, and let $F[\varphi_1] = F[\varphi_2] = 0$, where $\varphi_1 - \varphi_2$ is a function which does not change sign in the interval ab, and is different from zero only in the interval $a'b'$. Then there is a function φ_0, of the pencil determined by φ_1 and φ_2, and a value $\xi_0 (a' \leqq \xi_0 \leqq b')$, such that $F'[\varphi_0(x) \mid \xi_0] = 0$.*

In fact, if we write $F[\varphi_1 + \omega(\varphi_2 - \varphi_1)]$ as a function of ω, $F(\omega)$, it will be continuous in ω, and for a certain value $\omega = \omega_0$ will attain its maximum or minimum. Let this value of ω determine the function φ_0, and for the sake of definiteness, let us assume that $F(\omega) - F(\omega_0)$ is not positive if ω is in the neighborhood of ω_0. For the sake of definiteness also, let us assume that the functional derivative of F is positive when $\xi = b'$. Then it must be positive throughout the whole of the closed interval $a'b'$, unless it vanishes at some point of that interval, since it is continuous in ξ. Let us assume that it does not vanish.

In order to show the falsity of this assumption, let us construct the functions

$$
\begin{aligned}
\psi_{\omega, t}(x) &= 0 && (a \leqq x \leqq t,\ b' \leqq x \leqq b) \\
&= (x - t)\{\varphi_2(x) - \varphi_1(x)\} && (t \leqq x \leqq t + \omega) \\
&= \omega\{\varphi_2(x) - \varphi_1(x)\} && (t + \omega \leqq x \leqq b'),
\end{aligned} \tag{6}
$$

* This is essentially what Volterra uses as his postulate (IV). See loc. cit., p. 99 and p. 101. His postulate (I) is that if $\theta(x)$ is any variation of φ in h (not necessarily of one sign) in absolute value less than ϵ, then $|\Delta F/\epsilon h| < M$, uniformly.

and consider for what value of t, $t = \bar{t}$, the difference

$$F[\varphi_0(x) + \psi_{\omega,\,t}(x)] - F[\varphi_0(x)]$$

attains its upper limit, if ω is kept constant. We know that there will be such a value, since F is a continuous function of t, and we see directly that we can find ω' small enough so that if $\omega \leqq \omega'$ that value must be greater than a'. For if not, on account of the uniformity of the condition (III_1) and the resulting continuity of the derivative with regard to its functional argument, it would follow that we might take ϵ and h about a' so small that

$$F[\varphi(x) + \theta(x)] - F[\varphi(x)] > 0,$$

where $\theta(x)$ is any variation in the interval h, everywhere in absolute value less than ϵ,* and $\varphi(x)$ is any one of the functions $\varphi(x) = \varphi_0(x) + \psi_{\omega,\,t}(x)$. But this means that we could take ω small enough so that we should have $F[\varphi_0+\omega(\varphi_2-\varphi_1)] - F[\varphi_0] > 0$, which would be contrary to hypothesis.

Let us now take a series of values ω_n which approach zero as a limit. The corresponding series of values t_n of $\bar{t}$ has at least one limiting value, and any one of these limiting values, which may in particular be the point a', we may take as our ξ_0. This gives us our contradiction. For if $F'[\varphi_0(x) \mid \xi_0]$ were not zero, we could, owing to the uniformity of the condition (III_1), construct a function $\psi_{\omega_n,\,t}(x)$, with $t < t_n$, for which we should have

$$F[\varphi + \psi_{\omega_n,\,t}] > F[\varphi + \psi_{\omega_n,\,t_n}].$$

Our theorem is therefore proved.

6. The law of the mean is a consequence of this theorem in the same way as in differential calculus it is a consequence of Rolle's theorem.

LAW OF THE MEAN.† *Let $F[\varphi(x)]$ be a function for which (α), (II), (III_1) hold, and let φ_1 and φ_2 be two continuous functions in the given region, such that $\varphi_1 - \varphi_2$ does not change sign in the interval ab, and is different from zero only in the interval $a'b'$.*

* If $a' = a$, it is taken for granted that $\varphi_2(a) - \varphi_1(a)$ and $\theta(a)$ need not necessarily vanish. Also if $b' = b$, it is assumed that $\varphi_2(b) - \varphi_1(b)$ need not necessarily vanish.

† This theorem is due to Volterra (loc. cit., p. 103) who establishes it by means of formula (2), and thus as a consequence of the hypotheses (α) (I)–(IV).

Then there is a function φ_0 of the pencil determined by φ_1 and φ_2, and a value ξ_0 ($a' \leqq \xi_0 \leqq b'$) such that

$$(7) \quad F[\varphi_2(x)] - F[\varphi_1(x)] = F'[\varphi_0(x) \mid \xi_0] \int_a^b \{\varphi_2(x) - \varphi_1(x)\} dx.$$

7. We may now proceed, by means of these theorems and the hypotheses (α), (II), (III), to establish the formula (2). Let us consider first a continuous function $\psi(x)$ which does not change sign in the interval ab, and form the function $F(\omega) = F[\varphi_0 + \omega\psi]$. We shall endeavor to calculate $dF/d\omega$ for $\omega = 0$.

If we divide up the interval ab into parts $a_i a_{i+1}$, every one of which is in magnitude less than δ, we can write

$$F[\varphi_0 + \omega\psi] - F[\varphi_0] = \sum_{i=0}^{n-1} F[\varphi_0 + \psi_{\omega, a_i}] - F[\varphi_0 + \psi_{\omega, a_{i+1}}] + F[\varphi_0 + \omega\psi] - F[\varphi_0 + \omega\psi_{\omega, a}],$$

where $a_0 = a$ and $a_n = b$. But this is the same as the expression

$$\omega \sum_{i=0}^{n-1} F'[\varphi_0 + \psi_{\omega_i, a_{i+1}} + \gamma_i(\psi_{\omega, a_i} - \psi_{\omega_i, a_{i+1}}) \mid \xi_i] \times \int_a^b (\psi_{\omega, a_i} - \psi_{\omega, a_{i+1}}) dx + F[\varphi_0 + \omega\psi] - F[\varphi_0 + \psi_{\omega, a}],$$

where $|\gamma_i| < 1$ and $a_i \leqq \xi_i \leqq a_{i+1}$. We may for our purposes take all the intervals equal, and also $\delta = k\omega$ where k is any fixed number greater than 2. If then we take the limit of $\{F[\varphi_0 + \omega\psi] - F[\varphi_0]\}/\omega$ as ω approaches zero, we verify from the uniformity of assumption (III) and the results of § 4 that $dF/d\omega$ exists for $\omega = 0$, and is given by the formula

$$(8) \quad \left(\frac{dF}{d\omega}\right)_{\omega=0} = \int_a^b F'[\varphi(x) \mid \xi]\psi(\xi) d\xi.$$

Hence it follows that

$$(9) \quad \left(\frac{dF}{d\omega}\right)_{\omega=\omega} = \int_a^b F'[\varphi(x) + \omega\psi(x)]\psi(\xi) d\xi.$$

8. Consider now the general case where $\psi(x)$ is an arbitrary continuous function. We can write it as $\psi(x) = \psi_1(x) + \psi_2(x)$,

where

$$\psi_1(x) = \tfrac{1}{2}\{\psi(x) + |\psi(x)|\},$$
$$\psi_2(x) = \tfrac{1}{2}\{\psi(x) - |\psi(x)|\},$$

functions which do not change sign. Moreover if we consider the function $F[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2]$ we have the formulas

$$\frac{\partial F}{\partial \omega_1} = \int_a^b F'[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2 \mid \xi]\psi_1(\xi)d\xi,$$

$$\frac{\partial F}{\partial \omega_2} = \int_a^b F'[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2 \mid \xi]\psi_2(\xi)d\xi.$$

On account of the uniform continuity of F' over ξ and the two-parameter family of curves defined by ω_1 and ω_2, it follows that $\partial F/\partial\omega_1$ and $\partial F/\partial\omega_2$ are continuous functions of ω_1 and ω_2. Hence if we put $\omega_1 = \omega_2 = \omega$, we find that $dF/d\omega$ exists and is given by

$$\frac{dF}{d\omega} = \left(\frac{\partial F}{\partial \omega_1}\right)_{\substack{\omega_1=\omega \\ \omega_2=\omega}} + \left(\frac{\partial F}{\partial \omega_2}\right)_{\substack{\omega_1=\omega \\ \omega_2=\omega}},$$

which reduces to the form (9), since $\psi_1 + \psi_2 = \psi$. For $\omega = 0$, equation (9) reduces to equation (8), which holds for any continuous function $\psi(x)$, and is equivalent to (2).

From (9) we obtain (with Volterra) another law of the mean in the form

$$(10)\quad F[\varphi(x) + \psi(x)] - F[\varphi(x)] = \int_a^b F'[\varphi(x) + \theta\psi(x) \mid \xi]\psi(\xi)d\xi,$$

where $0 < \theta < 1$.

II. Other Methods of Deducing Formula (2).

9. We may obtain the formula (8) and hence the formula (2) from more general points of view. One point of view, which we shall consider, is closely related to the ideas which have just been developed; a second is a modification of the procedure of Fréchet.

For simplicity let us replace (α) by a new hypothesis (α'), the difference being that F is defined not only for all continuous functions in the given domain, but also for all functions in the given domain which have merely a finite number of discontinuities.*

* It is sufficient for what follows if we admit merely discontinuities *of the first kind*, so called (Lebesgue).

Let $\psi(x)$ be a continuous function, *of one sign*, and construct the functions ${}_\alpha\psi_\beta(x)$ defined as follows:

$$\begin{aligned} {}_\alpha\psi_\beta(x) &= 0 \quad (a \leqq x < \alpha,\ \beta < x \leqq b) \\ &= \psi(x) \qquad (\alpha \leqq x \leqq \beta). \end{aligned}$$

Let us consider besides (α'), the following postulates (β), (γ):

(β) $$\lim_{\omega=0,\ \alpha=a',\ \beta=b'} \frac{\Delta F}{\sigma} \text{ exists} \quad (a \leqq a' \leqq b' \leqq b),$$

where

$$\Delta F = F[\varphi(x) + \omega\,{}_\alpha\psi_\beta(x)] - F[\varphi(x)]$$

$$\sigma = \omega \int_\alpha^\beta {}_\alpha\psi_\beta(x)dx.$$

This limit will depend on $\varphi(x)$, $\psi(x)$, a' and b'; let us call it $G[\varphi(x), \psi(x) \mid a', b']$.

(γ) $G[\varphi(x), \psi(x) \mid a', b']$ is continuous in $\varphi(x)$.

10. If (α') and (β) are satisfied, we can verify directly that when a' and b' are equal, the function G is independent of $\psi(x)$, and we can therefore write

$$G[\varphi(x), \psi(x) \mid a', a'] = G[\varphi(x) \mid a'].$$

If F happens to have a functional derivative F', then

$$G[\varphi(x) \mid a'] = F'[\varphi(x) \mid a']. \tag{11}$$

If (α') and (β) are satisfied, we see also that if we let $|\alpha - a'|$, $|b' - \beta|$ approach zero with ω as functions of ω, then

$$\left(\frac{d}{d\omega} F[\varphi(x) + \omega\,{}_\alpha\psi_\beta(x)]\right)_{\omega=0} = G[\varphi(x), \psi(x) \mid a', b'] \int_{a'}^{b'} \psi(x)dx$$

if $a' \neq b'$, and

$$= 0$$

if $a' = b'$.

11. If (α'), (β), (γ) are satisfied, we can deduce, by making use of partial derivatives as in § 8, that

$$\begin{aligned} &G[\varphi(x), \psi(x) \mid a', b'] \int_{a'}^{b'} \psi(x)dx + G[\varphi(x), \psi(x) \mid b', c'] \\ &\times \int_{b'}^{c'} \psi(x)dx = G[\varphi(x), \psi(x) \mid a', c'] \int_{a'}^{c'} \psi(x)dx. \end{aligned} \tag{13}$$

And from this and the definition of $G[\varphi(x) \mid a']$ it follows that $G[\varphi(x), \psi(x) \mid a', b']$ is a continuous function of its two arguments a', b' in the region $a \leqq a' \leqq b' \leqq b$, and is hence uniformly continuous with respect to them. But these properties lead us to a formula equivalent to (8). For we have

$$\left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0} = G[\varphi(x), \psi(x) \mid a, b] \int_a^b \psi(x)dx,$$

and if we split the right-hand member into n parts, according to (13), and take the limit as n becomes infinite, remembering that G is uniformly continuous in a', b', we obtain the result

$$(14) \qquad \left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0} = \int_a^b G[\varphi(x) \mid \xi]\psi(\xi)d\xi,$$

where $\psi(x)$ is any continuous function which does not change sign. This again is generalized to hold for any continuous function $\psi(x)$, by the method of § 8.* Therefore, *formula* (14), *where $\psi(x)$ is any function continuous $a \leqq x \leqq b$, is a consequence of postulates* (α'), (β), (γ).†

12. It is now evident what sort of restriction is sufficient in order that formula (1) may reduce to (2). In fact we can deduce (14) from (α') and the two following hypotheses:

(β') Let $\psi(x)$ be any limited function, continuous except for a finite number of discontinuities (of the first kind); then we assume that

$$\left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0}$$

exists and is distributive with respect to ψ, i. e.,—

$$\left(\frac{d}{d\omega} F[\varphi + \omega(\psi_1 + \psi_2)]\right)_{\omega=0} = \left(\frac{d}{d\omega} F[\varphi + \omega\psi_1]\right)_{\omega=0} + \left(\frac{d}{d\omega} F[\varphi + \omega\psi_2]\right)_{\omega=0}$$

* If in the definition of $G[\varphi(x), \psi(x) \mid a', b']$ we do not allow ω to change sign as it approaches zero, we are led to two functions $G_+[\varphi(x) \mid \xi]$ and $G_-[\varphi(x) \mid \xi]$, whence, instead of (14) for the general ψ, we have

$$(14') \quad \left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0} = \frac{1}{2}\int_a^b \{G_+[\varphi(x) \mid \xi](\psi(\xi) + |\psi(\xi)|) + G_-[\varphi(x) \mid \xi](\psi(\xi) - |\psi(\xi)|)\}d\xi.$$

† If $\varphi(x) = \Phi_1(x)$, $\psi(x)$ cannot be negative; if $\varphi(x) = \Phi_2(x)$, $\psi(x)$ cannot be positive.

(γ') The function $G[\varphi(x) \mid \xi]$ defined in (β) exists for $a \leqq \xi \leqq b$.

The postulate (β') is substantially the hypothesis used by Fréchet with (α) in defining the existence of a differential.* Moreover if we restrict ourselves to a function $\bar{\psi}(x)$ of one sign, we see that the function

$$G'[\varphi(x), \psi(x) \mid a', b'] = \lim_{\omega=0} \frac{F[\varphi(x) + \omega_{a'}\psi_{b'}(x)] - F[\varphi(x)]}{\omega \int_{a'}^{b'} \psi(x)dx}$$

exists, provided that $a' \neq b'$, and from (β') we see that the relation (13) holds with G' for G. And if we define

$$G'[\varphi(x), \psi(x) \mid a'a'] = G[\varphi(x), a'],$$

it follows as in (11) that G' is uniformly continuous in a' and b' in the region $a \leqq a' \leqq b' \leqq b$. Hence we obtain again formula (14) for a function ψ of one sign, and, finally, for any function ψ if ψ is limited and has merely a finite number of discontinuities (of the first kind).

13. The advantage of the last set of assumptions (α'), (β'), (γ') is that they do not demand the continuity of $G[\varphi(x) \mid \xi]$, or $F'[\varphi(x) \mid \xi]$ if it exists, with respect to the functional argument $\varphi(x)$. If for a given $\varphi(x)$, we suppose that $G[\varphi(x) \mid \xi]$ does not exist for certain special values of ξ, we have the case that F depends in a special manner on $\varphi(x)$ for those values of x.† The case where $A[\varphi(x) \mid \xi]$ in (3) is discontinuous comes under this specification. The property of possessing a functional derivative in general, however, seems to depend on fundamental properties of continuity with respect to aggregates of functions, the study of which is thereby rendered specially inviting.

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* M. Fréchet, loc. cit., p. 141. In order to justify the substitution of a postulate akin to (α') rather than (α) see F. Riesz, "Les opérations fonctionelles linéaires," *Annales scientifiques de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 2, who shows that a linear relation such as that expressed in the conception of differential can always be extended by definition to apply to certain classes of discontinuous arguments.

† V. Volterra, loc. cit., p. 144.

NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Geometry. By D. M. Y. SOMMERVILLE. London, G. Bell and Sons, 1914. 16mo. xvi+274 pp.

FEW recent writers upon non-euclidean geometry have approached their task with better chances of success than attended Dr. Sommerville in the preparation of the present volume. Anyone who has seen his scholarly and painstaking "Bibliography of Non-Euclidean Geometry"* will realize that in so far as a knowledge of what others have written upon a subject is a desirable qualification, the present author was most fortunately placed. Furthermore he is the happy possessor of an excellent literary style. A book written by such a writer should be interesting and stimulating; the present book has both of these characteristics. The choice of material is admirable, and the narrative continually illumined by historical notes.

When the fairies were invited to the christening of the Sleeping Beauty one of the sisterhood was unfortunately overlooked, and her absence caused all the trouble that came afterwards. So here, one thing is lacking, singleness of aim. Says the author (page vii):

"It is hoped that the book will prove useful to the scholarship candidate in our secondary schools who wishes to widen his geometrical horizon, to the honours student at our universities who chooses geometry as his special subject, and to the teacher of geometry in general who desires to see how far strict logical rigour is made compatible with a treatment of the subject matter capable of comprehension by schoolboys."

Does not this programme spell "failure" from the start? Complete rigor and a treatment comprehensible by schoolboys, even by Scotch ones, who indubitably work harder and know more than Americans of like age, are so far incompatible that it is quite useless to make the attempt. The needs of the schoolboy and of the candidate for honors are so different that a book intended for both will suit neither. In the present work if we confine ourselves to the first four chapters,

* London, Harrison, 1911. Reviewed in the BULLETIN, vol. 18, Feb., 1912.

which the author declares (page vi) constitute the rudiments of the subject, and omit all material in fine print, we still find such topics as the exponential function with a complex argument, hyperbolic functions, imaginary points and lines, the differential equation

$$\frac{d^2a}{dr^2}+\frac{a}{k^2}=0,$$

the bizarre integral

$$\int_0^\infty k \sin\frac{c}{k}\, dAdc,$$

the plane at infinity, and the desmic configuration. We have yet to find any actual schoolboy who was able to make very much of such expressions. But if we consider the book from the point of view of the "honours student" we observe that he will find a romantic optimism in many statements and proofs altogether at variance with what he is learning in his analysis and his algebra. The author's disclaimer (page vi) of any attempt to make the book rigorously logical with a detailed examination of all assumptions is no sufficient excuse.

These are surely serious charges to bring against any author, especially against one so well equipped as is Dr. Sommerville. Let us try to sustain them in detail, without losing sight of the various attractive features which the book surely possesses.

The first chapter is purely historical and deals with the landmarks in the history of non-euclidean geometry, much as they are described in a score of books. Historical or not, the author is true to his British didactic instinct and closes this chapter, like all the others, with a number of examples for the student to work out. Surely the British text-book writers lead the world in this respect. Says Cremona in the preface to the English edition of his Geometria proiettiva:*

"Unless I am mistaken the preference given to my Elements over the many treatises on modern geometry published on the continent is to be attributed to the circumstance that in it I have striven, to the best of my ability, to imitate the English models. . . . I aimed therefore at simplicity and clearness of expression, and I was careful to supply an abundance of examples of a kind suitable to encourage the beginner."

If previous writers of text-books on non-euclidean geometry have omitted such examples, was it from conviction or laziness?

* Second edition, Oxford, 1893, p. xiii.

In Chapter II we begin the systematic treatment of the hyperbolic plane. The writer says little about axioms, except to give Hilbert's classification into axioms of connection, order, congruence, continuity, and parallelism. It is by no means easy to say how the author wants us to treat these axioms. The inference is that we are to accept all save the last, for we read (page 27): "We shall assume as deductions from them the theorems relating to the comparison and addition of segments and angles," and two pages later we have Pasch's axiom that a line which meets a side of a triangle and a second side produced meets the third side. But this axiom depends for its statement upon the axioms of order, i. e., the axioms of an open order. Yet if we accept an open order at the outset why trouble ourselves at all about the elliptic plane where a straight line has a closed order and where Pasch's axiom may not be true? The fact is that the whole axiom question is beset with difficulties. If a writer who has not had the needful special training undertake to make up his own set of axioms, he is likely to make a botch of it; if he accept uncritically a set which some one else has developed, he is in grave danger of running into contradictions.

"Revenons à nos moutons." The first dozen pages of Chapter II go to a discussion of parallel lines in the hyperbolic plane and give the important theorems in good form. All is clear and well defined. The first break occurs on page 39 where we read:

"*Two parallel lines can therefore be regarded as meeting at infinity, and further the angle of intersection must be considered as being equal to zero.*"

We find further on (page 46):

"We shall extend the class of points by including a class of fictitious points called points *at infinity*. These points function in exactly the same way as ordinary, or, as we shall say, actual points. . . . On every line there are two points at infinity."

Our comment on these statements is as follows: We only know two ways of extending the class of points to include new members. We may define the new points by means of already recognized figures, as for instance, we might define a "point at infinity" as the totality of lines parallel to a given line and to one another, or, secondly, we might define a "point at infinity" by a set of postulates as

There exists a class of P.I.'s.

Each line contains two P.I.'s.
Each point and each P.I. determine a line.
Every two P.I.'s determine a line.

Starting with either of these methods we may go on to define lines and planes at infinity, and then similarly ultra-infinite points, lines and planes. Either plan is permissible, neither is entirely simple. But when our present author tells us (page 48) that a bundle of lines perpendicular to a plane *have* an ideal vertex, and further that "*ideal points* thus introduced behave exactly like actual points" we are left wondering. They surely do if we confine ourselves to projective properties, but if, perchance, we seek the distance from an actual to an ideal point we are in very serious difficulty.

There are two other points to be noticed while we are upon these thorny pages of the book. We read (page 41):

"Thus the distance between the two given lines AB' and LX first diminishes and then tends to infinity. It must therefore have a minimum value." This is, of course, a pure assumption and should either be proved, or made explicitly. Then we read in the note to page 46:

"The definition of a *conic* which we shall use is 'a plane curve which is cut by any straight line in its plane in two points.' For the explanation of the case of 'imaginary intersections' see Chapter III, § 5."

Here, if we overlook the removable objection that a tangent meets a conic in but one point, we still wonder what is the author's definition of a curve. If he mean an analytic curve, the definition for a conic is entirely proper. For if we take the origin upon such a curve and the axes parallel to the asymptotes, the abscissa and ordinate of every other point will be analytic functions of the slope of the line from the point to the origin, and these functions are single valued, and have single valued inverses. Hence we may express the curve in the form

$$x = \frac{al + b}{cl + d}, \quad y = \frac{a'l + b'}{c'l + d'}, \quad l = \frac{y}{x},$$

and we have the usual conic of commerce. But in the present book there is, up to the present point, no machinery for an exact statement of this sort and as for the method of introducing imaginaries, well, we shall deal with that presently.

The last part of the chapter goes to the development of

trigonometry following the classical methods of Lobachevsky which depend upon the parallel angle. Here everything is elementary, though at times a bit involved. Moreover, it is stated upon page 58 that the solution of

$$f(x + y) = f(x)f(y)$$

must be $f(x) = c^x$. This is true if, and only if, the restriction be imposed that $f(x)$ shall be a continuous function not $= 0$.* Our author can not have been ignorant of the necessity for this restriction, since on page 114 in discussing the corresponding equation for $\cos x$ he mentions the requirement of continuity. Query, how many schoolboys will understand what a functional equation means anyway? The chapter ends with an ingenious discussion of the relation between the defect of a triangle and its area.

Chapter III is given to elliptic geometry. This must be worked out "ab initio" since the methods of hyperbolic geometry, based on the parallel angle, are inapplicable. A curious omission occurs (page 89) where the author recognizes that if a straight line be a closed curve, two coplanar lines might intersect once or twice, but fails to show that they might not have more than two intersections. As a matter of fact the proof is a bit delicate. He rightly appreciates that the most attractive feature of elliptic geometry is the perfect duality between point and plane, angle and distance. To bring this out he needs projective geometry, and since he deems it unwise to assume a knowledge of the latter on the part of his readers he gives a short introductory account of the subject on pages 93–98. This is no easy task. The idea of a cross ratio is fundamental in projective geometry, but the elementary definition in terms of the ratios of certain segments is invalid in non-euclidean space, while (wrongly, as we believe) he considers that a definition based upon successive quadrangle and quadrilateral constructions is too difficult or abstract. He therefore adopts the following unhappy expedient. We read (page 94):

"If two ranges of points are made to correspond in such a way that to every point P on one range corresponds uniquely a point P' on the other and *vice versa*, the ranges are said to

* Conf. Wilson, "Note on the function satisfying $\phi(u)\phi(v) = \phi(u + v)$" *Annals of Mathematics*, series 2, vol. 1, 1899, and Darboux, "Sur le théorème fondamental de la géométrie projective," *Math. Annalen*, vol. 17 (1880).

be *homographic*," and on page 95: "It can be proved that two homographic ranges can always be connected by a finite number of projections and intersections." This familiar statement has always reminded us of the sea-serpent. It is based upon an illusion, but is of venerable antiquity, is believed in by many, and is indestructible. Yet there is no difficulty in showing its falsity. Let us ask the writer in what domain we are supposed to be. If the answer be the real domain, since nothing yet has been said about imaginaries, we ask him how many projections and intersections are necessary to accomplish the transformation

$$x' = x^3.$$

If, however, we are in the complex domain, we have the classical example

$$x' = \bar{x},$$

i. e., each point of a line corresponds to its conjugate imaginary. Of course the statement is true if the domain be complex, and the transformation analytic, but these are just the restrictions which are usually overlooked.

The imaginary first appears explicitly on page 97 where we read, in discussing involutions: "If two real self-corresponding points do not exist we introduce by definition conjugate pairs of 'imaginary' points, much in the same way that ideal points were introduced in hyperbolic geometry." With this epigrammatic statement the author leaves the matter; he now feels just as free to use imaginary points as real ones.

It is fair to say that this happy solution of the imaginary difficulty is not new. The principle, so far as we understand it, is this. If by a continuous change we can make two real points coalesce and then disappear we are free to say that they have been replaced by two imaginary points, and repel with scorn all inquiries as to the nature of these latter. Let us make a careful drawing of the curve.

$$(x^2 + y^2 + 1)^{1492}(y - x^3) + \epsilon = 0,$$

where ϵ is microscopic. This curve will lie extremely near to

$$y = x^3.$$

Some lines will meet our curve in three real points, others in only one. Shall we, therefore, say that the latter meet the

curve in one real and *two* imaginary points? If our pupils be too young to comprehend the truth in these matters, let us, at least, tell them nothing else.

A slight slip of a different sort occurs on page 99, where we read "Similarly, in three dimensions a polar system determines a surface of the second degree, or quadric surface." The author has clearly forgotten the null system. The remainder of the chapter is devoted to paratactic lines, Clifford surfaces, and trigonometric relations. The subject is interesting and the treatment good, although we find on page 111 the erroneous statement "*all the common tranversals of two right paratactic lines are left paratactic lines.*"

The fourth chapter is given to analytic geometry. A point in the plane is determined by its Weierstrass coordinates, namely

$$x = k \sin\frac{r}{k}\cos\theta, \quad y = k \sin\frac{r}{k}\sin\theta, \quad z = \cos\frac{r}{k},$$

where r is the distance from the origin, θ is the angle between the radius vector and the positive half of the X axis, and k the space constant. The subject matter is beset with fewer pitfalls than that which preceded, and the handling is satisfactory. The chapter closes with the account of two configurations, the triangle configuration of Desargues and the desmic tetrahedron configuration of Stéphanos.

A distinct change comes over the book with Chapter V. This excellent chapter is given to various representations of non-euclidean space in euclidean space. Three such representations are discussed. First we have the projective representation, and Cayley's projective metric; secondly the geodesic representation where the non-euclidean plane is developed on a euclidean surface of constant curvature, and lastly the conformal representation where non-euclidean lines appear as euclidean circles orthogonal to a fixed circle. Our only criticism is the mild wish that the chapter had been a little longer, and that the author had pointed out more clearly that in some cases we are able only to develop a part of our space.

Our author has next the curious idea of interpolating another philosophical chapter, to bring the history of non-euclidean geometry to date, and to discuss an interesting philosophical question. These are rather deep subjects to be

disposed of in a twenty page chapter, and the treatment of the modern abstract view of geometry is scant. The particular philosophical question is this: is our space of experience euclidean or non-euclidean? The author asks (page 203):

(1) Can the question of the true geometry be settled a posteriori or experimentally?

(2) Can it be decided on philosophical grounds?

(3) Is it, after all, a proper question to ask, one to which an answer can be expected?

Taking these up in turn he first shows that if the space constant be not infinite, it is so large that we shall never make any serious error in assuming that our space is euclidean. The second question is easily disposed of: no one believes nowadays that we can settle the question of parallels a priori. The third idea, that the question is improper, and that we can no more say that the parallel axiom is true or false, than that the metric system is true or false has the great authority of Poincaré, and we believe that Dr. Sommerville inclines to a like opinion. He puts the case as follows (page 209):

"A further point—and this is the 'vicious circle' of which we spoke above—arises in connection with the astronomical attempts to determine the nature of space. These experiments are based upon the received laws of astronomy and optics, which are themselves based upon the euclidean assumption. It might well happen then, that a discrepancy observed in the sum of the angles of a triangle could admit of an explanation by some modification of these laws, or that even the absence of any such discrepancy might still be compatible with the assumptions of non-euclidean geometry."

This is rather a fascinating idea. It is evident, for instance, that if light were not propagated along straight lines, it would be hard to determine the parallel question astronomically. However, leaving aside the minor doubt as to just how far our science of physical optics *is* dependent upon the parallel postulate, the root of the difficulty seems to us that the argument is a little too strong. Could we not render all scientific deduction impossible by similar considerations? If we make any experiment whatever, the results can be interpreted in a transfinite number of ways and the best that we can hope for is to find what conclusions are compatible with various accepted hypotheses. Let us take a simple question. Are there any mountains on the other side of the moon? At

present it seems hard to find a crucial experiment to settle the matter. Suppose, however, that in the course of centuries we set up some sort of wireless telegraphy with those eminent engineers who superintend the construction of canals on Mars, and put the question to them. We may well imagine their replying "We do indeed observe there markings which are in every way similar to those markings on the earth to which you give the name of Himalayas." What conclusion should we then draw? It might be that the laws of optics for the Martians were different from our own, it might be that whereas rays of light reflected from the earth travelled in straight lines those reflected from the other side of the moon pursued spirals. Endless other conjectures are possible. The obvious conclusion is, however, that there are such mountains, and that no vicious circle was involved in asking the question.

The seventh chapter is devoted to circles, and in fact, the amount of attention paid to circles and circle transformations is a distinguishing characteristic of the book. The treatment is greatly clarified by the use of a transformation first given analytically by Hausdorf* and later synthetically by Liebmann.† Confining our attention to the real and actual domain of the hyperbolic plane, we take a ground plane F. Then if E be any other actual plane, its intersection with the absolute quadric will be orthogonally projected upon F into a circle, if we include under this head not only circles with actual centers, but horocycles and equidistant curves. Conversely, every such curve can be reached in this way. This being shown, our author says (page 230):

"Hence there is a (1, 1) correspondence between the circles in a plane and the planes in space."

"Ahimé!" before saying this why did he not try to run his machine backwards? Had he done so, instead of taking Liebmann's dictum, he would have perceived that two planes which lie symmetrically with regard to F give the same circle therein, so that the correspondence is not (1, 1) but (1, 2). Hausdorf, proceeding analytically, points this out with care‡ and subsequently changes it to a (1, 1) correspondence by covering the plane over with two different layers of points.

* "Analytische Beiträge zur nicht-euklidische Geometrie," *Leipziger Berichte*, vol. 51 (1899).

† "Synthetische Ableitung der Kreisverwandtschaften, etc.," ibid., vol. 54 (1902).

‡ Loc. cit., p. 177.

It is curious that this mistake was not brought to our author's notice by the following inconsistency. He points out (page 218) that two circles may intersect in four (real) points, the resulting angles being equal in pairs, but warns us at the top of the next page that the four angles are not, in general, equal. We then reach this interesting theorem (page 231):

The angle between two planes is equal to the angle of intersection of their marginal images.

The marginal image of a plane is the circle which corresponds in the present scheme. Of course the correct theorem is this:

Given two circles in F, the one represented by the planes E_1E_2, the other by G_1G_2. Two angles of these circles will be equal to the angle of the planes E_1G_1 (or E_2G_2), the other two will be equal to the angle of the planes E_1G_2 (or E_2G_1).

Chapter VIII discusses circular transformations both geometrically and analytically. Part of the analytic work seems a bit messy; the geometrical treatment is ingenious, especially the proofs of the theorems that every circular transformation of the non-euclidean plane is conformal, that it can be factored into the product of a congruent transformation and an inversion or radiation, and that every circular transformation of the plane corresponds to a congruent transformation of space. These are valuable theorems which we are glad to see in such a work. Unfortunately we must note one more oversight, analogous to those which marred the chapter preceding, and common to most writers upon this subject.*

Let us take what our author calls (page 241) a *hyperbolic inversion.* Here corresponding points P and P' are collinear with the center of inversion O, the product of the hyperbolic tangents of the halves of their distances from O is constant, and they *are on the same side of O.* The analogy to euclidean inversion seems complete, for here we may define inverse points as being collinear with O, the product of their distances therefrom is constant, and *they are on the same side of O.* There is, however, an important distinction. The requirement that the two points should be *on the same side of O* is not

* We make an exception in favor of a really careful article by Beck, "Ein Seitenstück zur Möbiusschen Theorie der Kreisverwandschaften," *Transactions Amer. Math. Soc.*, vol. 11 (1910). Apparently Dr. Sommerville was not familiar with this article, at any rate there is no mention of it in his bibliography. Beck's circles are oriented, and treated as the envelops of oriented straight lines. His paper might equally well have been called a Seitenstück to Laguerre's "Géométrie de direction."

an analytic requirement. In the euclidean plane it is unessential, for we may also define inverse points as being collinear with O and conjugate with respect to the circle of inversion. In the non-euclidean plane we have not this refuge, and, analytically speaking, a point in general position has *two inverses*. We bring out this important fact as follows.

We may map the upper half of the euclidean plane ($y>O$) upon the actual domain of the hyperbolic plane in such a way that the correspondence is one to one, conformal, and carries circles to circles; i. e., a circle in the hyperbolic plane will correspond either to a circle in the euclidean plane, or to a part of such circle and the reflection of the rest in the x axis. Let P be a point of the upper euclidean half-plane and c_1 any circle entirely in that half-plane. Let P' and c_1' be the corresponding point and circle of the hyperbolic plane. Then the inverses of P' in c_1' will be the points which correspond to the inverse of P in c_1 and that which corresponds to the reflection in the x axis of the inverse of P in the reflection of c_1. We may go a step further in this direction, and announce a theorem which is, perhaps, new.

The only analytic circular transformations of the non-euclidean plane which are single valued are congruent transformations.

It is ridiculously easy to prove this analytically, but perhaps it will be more sportsmanlike to follow the methods of the book before us. Since our transformation carries a point to a point, a circle to a circle, and the points of a circle to points of a circle for our plane F, it is (page 237) a conformal transformation. Hence a circle orthogonal to itself will go into another such, i. e., a null circle into a null circle. In space we have a transformation of plane to plane, which carries a plane tangent to the absolute into another such plane, in fact a congruent transformation of space (page 238). Since the transformation of F is to be one-to-one, two planes tangent to the absolute which correspond to the same null circle in F, i. e., two planes tangent to the absolute whose common line is in F, will correspond to two other such planes. Here a line in F is carried into a line in F, i. e., our congruent transformation carries F into itself.

The concluding chapter goes to a classification of non-degenerate conics under the non-euclidean congruent group. The subject matter is not particularly novel, and the treatment calls for no special comment.

If we were to pass a final judgment upon the book before us it would be this: It is in some respects an improvement upon its predecessors; it is by no means the best book that the present author could have written.

J. L. COOLIDGE.

CAMBRIDGE, MASS.,
December, 1914.

MINKOWSKI'S WORKS.

Gesammelte Abhandlungen von Hermann Minkowski. Herausgegeben von D. HILBERT. Leipzig, B. G. Teubner, 1911. Band 1, xxxi + 371 pp. Band 2, iv + 465 pp. Two portraits.

MINKOWSKI's work divides itself naturally, and his collected works are divided, into four parts: Theory of quadratic forms, 242 pages, Geometry of numbers, 230 pages, Geometry, 180 pages, Physics, 163 pages. In addition to this the volumes before us contain the author's address on Dirichlet, 15 pages, and Hilbert's commemorative address on Minkowski, 27 pages. This heartful and touching tribute of a life-long friend and fellow-worker is in reality also a critical review of Minkowski's great achievements in mathematical science, and it may be that the best thing for us to do in reviewing these volumes would be to follow the example of the reviewer in another Bulletin* and translate the chief portions of that address. The availability of the address in the original, where it should be read as a whole, and, in abstracts, in French makes repetition here seem really unnecessary.

We are accustomed to precocious exhibitions of genius in mathematicians, and we often cite the case of Galois, who died in his twenty-first year after accomplishing work of which the fundamental importance was not and perhaps could not be appreciated until a much later date. Minkowski in his eighteenth year submitted to the Paris Academy a memoir on quadratic forms with integral coefficients which fills 142 pages of his collected works and which received the Grand Prix des Sciences mathématiques. Measured in pages, one-sixth of Minkowski's work was written before he was 18. His work of the next ten years deals almost exclusively with quadratic forms.

* *Bull. Sci. Math.*, France, vol. 36 (1912), p. 73.

In the early nineties, when about twenty-eight years old, Minkowski is found opening up his great work on the geometry of numbers, which may be said to occupy about a decade of his life. His book, Geometrie der Zahlen, of which the first part appeared in 1896 in the midst of this decade, was never finished. (The short second part published by Hilbert after the author's death was really merely the conclusion of the fifth section of the first part.) It is to the geometry of numbers with its fundamental concept of convex bodies that we may probably attribute the greatest brilliancy of Minkowski's brilliant career; and to it, also, we may look for his most permanent impressions upon mathematical thought. His work on geometry is a natural corollary of that on the geometry of numbers, and the major part of it in his collected works is the hundred page memoir, previously unpublished, on the theory of convex bodies and in particular on the foundation of the surface-concept as applied to them.

At the close of his life, Minkowski was working on physics. He had printed in the Encyclopedia the article on Capillarity, which is a model of exposition. He had printed his memoir on electromagnetic phenomena in moving bodies, had delivered his address at Cologne on Raum und Zeit, and was preparing a contribution on the deduction of the electromagnetic equations for moving bodies from the point of view of electron theory. This last contribution prepared by M. Born, more from his reminiscences of conversations with the author than from Minkowski's unintelligible notes, is included in the collected works.

Just how much praise the future will attribute to this work on electrodynamics we cannot estimate. The posthumous contribution seems to have had little influence; the preceding memoir is written in a somewhat clumsy notation, appears at times to be translating rather blindly known results into a new notation, and contains some errors.* But the Cologne address is a gem. In it is formulated with satisfactory simplicity the proposition that the laws of physics are fundamentally relations between certain vectors or other geometric elements (affected by coefficients) attached to four-dimensional loci and that the laws which we observe are relations between the projections of those vectors or other ele-

* Wilson and Lewis, "Relativity," *Proc. Amer. Acad.*, vol. 48, p. 495 (1912).

ments on our space-time system of reference. This idea was at the basis of the work in the memoir on electromagnetic phenomena, but needed the address to emancipate it.

There is one point in Minkowski's work which is of negligible importance for that work but which has attracted such comment that it is worth discussing for itself. When introducing his matrical calculations, the author appends a footnote:* "Mann könnte auch daran denken, statt des Cayleyschen Matrizenkalküls den Hamiltonschen Quaternionenkalkül heranzuziehen, doch erscheint mir der letztere für unsere Zwecke als zu eng und schwerfällig." †

Silberstein in published papers and in his book on relativity has shown conclusively that the analytical work on relativity can be carried out with extreme simplicity of notation by the use of quaternions. The most casual comparison of Silberstein's analysis with that of Minkowski will reveal very strikingly the neat quaternion and the clumsy matrix. The possibility of using quaternions lies in the fact that the Lorentz group is a group of rotations (imaginary or non-euclidean) in the four-dimensional manifold of x, y, z, t, and that, as Cayley showed, quaternions may be used to determine four-dimensional rotations by the formulas $q' = QqQ$, $T^2Q = 1$, where q denotes position.

The question, however, remains whether quaternions, though neat, are really appropriate. That they are not is indicated by Silberstein's admission that he had tried a whole year in vain a great variety of quaternion operations for relativistic purposes before discovering Cayley's proposition.‡ But there is a more fundamental reason: A four-dimensional vector is not a quaternion. A four-dimensional vector becomes a quaternion only after the choice of a second (reference) vector which is the scalar axis. Thus in four-dimensional analysis, just as in three dimensions, a quaternion involves two vectors, though in a very different way. The use of quaternions in geometric analysis in four dimensions involves, therefore, an extraneous element, just as the use of cartesian coordinates in geometric analysis in a plane involves elements extraneous to the geometric problem.

* Vol. 2, p. 375.

† L. Silberstein, Relativity, viii+295 pp., Macmillan, 1914, states on page 150: "Minkowski himself despised Hamilton's calculus of quaternions as 'too narrow and clumsy for the purpose' in question."

‡ *Phil. Mag.*, May, 1912, p. 790.

When using quaternions we have to be very careful to distinguish results which are intrinsically geometric from results which are relative to the direction of reals. This may be illustrated from the theory of complex numbers. The transformation $z' = az + b$, where a, b, z, z' are vectors in a plane (which become complex numbers after the choice of a real axis), is a transformation of similitude, no matter what direction be chosen as the axis of reals, but the transformation is not independent of that choice.

In relativity the time axis is accidental to a particular observer or group of observers and should be chosen *after* the fundamental work is done, not before. The analysis which is really appropriate to the theory of relativity as conceived by Minkowski is Grassmann's. Even a vector analysis (such as that used by Lewis and me, loc. cit.) assumes an origin, which is theoretically "de trop," though practically not much in the way. Is it not unfortunate that Minkowski should have followed the English Cayley, referred to the Scot-Irish Hamilton, and ignored the German Grassmann? Should not some Geheimer Regierungsrat among his colleagues have given him secret directions to avoid such an unpatriotic scientific mésalliance?

E. B. WILSON.

SHORTER NOTICES.

Histoire des Mathématiques. Par CH. BIOCHE. Paris, Belin Frères, 1914. vi+93 pp. Price, 1 fr. 75 c.

IT is one of the strange anomalies in the making of books that France, where the best work in the history of mathematics was done in the eighteenth and early nineteenth centuries, should have done so little in this line in recent years. Montucla, who wrote the first interesting general history of the subject; Delambre and De la Lande who were his worthy successors; Bossut, whose style maintained well the earlier traditions; Libri, writing in France although Italian by birth, and writing with the style of a novelist; Chasles, putting more mathematics into his work than his predecessors,—all these men contributed very notably to the appreciation of the historical development of the science, and set a high standard of style if not always of scholarship. But of late France has produced no general histories of mathematics worthy the name.

To be sure Paul Tannery was a remarkable scholar in his field, and there are few men now living who can rank with Duhem, but the former never attempted a general history and the latter has not done so as yet.

It is for such reasons as these that one looks with special interest to even the simplest efforts to revive the splendid traditions of France in a field that was one time peculiarly her own. And so, while the work of M. Bioche is modest in size and humble in purpose, it is none the less welcome as an evidence of growing interest in the subject. Urged as he says by his " excellents camarades d'Ecole normale, Henri Bergson et Gaston Milhaud," M. Bioche has set about to write a history of ideas rather than one of literature, a record of the stream of progress of mathematics rather than a biography of mathematics.

As a result he has produced a manual somewhat like Mr. Rouse Ball's little Primer of Mathematics in size, although quite different in general treatment. The work consists of eleven chapters, devoted successively to the following topics: Mathematics before the time of the Alexandrian school; the school of Alexandria; the middle ages; the geometry of the Renaissance; the origin of algebra; analytic geometry; the infinitesimal calculus; geometry in the seventeenth and eighteenth centuries; the nineteenth century; ancient astronomy; modern astronomy.

It is not to be expected that new contributions should appear in such a handbook, and there is nothing of this nature to record. A few assertions may be found, however, which are not usually seen in elementary treatises, as that Archytas of Tarentum was probably the first to consider a curve of double curvature, that Aristarchus was the first to make a tentative evaluation of the elements of the solar system, that Apastamba stated the pythagorean theorem before the conquest of Alexander, that Oresmus had the true idea of function in his De latitudinibus formarum, and that the early Greeks used the idea of the climate (the parallel of equal maximum day lengths) instead of the notion of latitude with respect to the equator.

The style of the author maintains the high reputation of the French school and leaves little to be desired. The statements are, however, not altogether free from error, and the omission of a name like that of Mahavir is difficult to explain. The

dates are generally stated as if they were known with certainty, as that Euclid was born in 330 B.C., and Fibonacci in 1175 A.D., while in reality many of these statements are very doubtful and are liable to be put to unfortunate use by the novice. Among the probable errors of statement are the assertion that Heron was a contemporary of Hipparchus, and that Jordanus Nemorarius was the Jordanus who was general of the Dominicans. Among the certain errors are the assertions that Alcuin was abbot of Canterbury, and that Omar Khayyám was of Arab rather than Persian stock; and among the typographical errors are the printing of Gunther for Günther (page 26), Muller for Müller (page 30), Harriott for the preferred form of Harriot (page 34), and Plucker for Plücker in the index (with a wrong reference). But in spite of these little blemishes the book will serve a good purpose, particularly among the students of the secondary schools of the French-speaking countries. DAVID EUGENE SMITH.

Solid Geometry. By SOPHIA FOSTER RICHARDSON. Boston, Ginn and Company, 1914. iv + 209 pp.

As the author states in the preface, she gives in this book the "usual course in solid geometry more complete in logical structure than that of the text-books commonly used." Definitions and axioms are quite numerous and prominent and it is by carefully stating these that many difficulties are avoided. For instance there is no difficulty nor incompleteness in the proofs of the theorems about the intersection of a cylinder or cone with a plane through an element and another point of the surface because the theorems are explicitly limited to convex surfaces. We find here also the practice, too rare in American texts, of establishing the existence of a geometric object before defining it. Thus the theorem that a straight line perpendicular to each of two intersecting straight lines at their point of intersection is perpendicular to every straight line in their plane passing through their point of intersection, is given before the definition of a perpendicular to a plane. Similarly the theorem "Any tangent line to a convex cylindrical surface and the element through its point of contact determine a plane which contains no other point of the surface" leads to the definition of a tangent plane to a convex cylindrical surface. As in most texts, geometric locus is defined and the two parts of a locus problem are pointed out,

but the notable fact is that the author practices what she preaches, not only in showing that the intersection of two planes is a straight line but also in various other theorems in which too many authors are satisfied with half proofs.

Several theorems on parallel lines and planes are given before theorems on perpendicular lines and planes, an arrangement which would seem to offer less resistance to the beginning student. The exercises throughout the book demand thought, most of them being theorems rather than numerical problems.

The last chapter of the book on "Symmetry and similarity" and the Appendix on "Irrational numbers, variables and limits" offer an elementary treatment of topics to which the student of solid geometry does not usually have access.

RAINARD B. ROBBINS.

A Geometrical Vector Algebra. By T. PROCTOR HALL. Western Specialty, Vancouver, B. C. 30 pp.

HALL'S Geometric Vector Algebra is hardly what its name implies, since trigonometry and determinants are constantly needed to help out the computations of the vector algebra. We fail to attain the promise of the introduction, where we are led to expect a calculus of the space elements themselves. The exposition of principles in the first nine pages is fairly sound, although fragmentary. The word "tensor" is not defined, but means "length of a vector." S_{ab} and V_{ab} mean, substantially, the dot and cross products of Gibbs. But the product of two vectors is defined as a new vector, which turns out to consist of the usual vector product plus another vector term having the direction of the multiplying vector and the length S_{ab}. Evidently with this multiplication not much progress can be made toward a working system. The author himself proves that multiplication is not commutative, associative, nor, in general, distributive. The square of a unit vector, by the definition, is equal to the vector itself, but as a perpendicular operator the square is negative unity; whence equals cannot be put for equals.

Trouble comes in the attempt to show that quaternions are special cases of these operations. The first three fundamental characters of a quaternion on page 10 are correctly stated, but the fourth does not hold, for it contradicts the proof on page 6 that an operator is not distributive. We may indeed resolve the vector into components, but to say that, as an operator,

the vector can be expressed as the sum of three parts, but cannot be distributed with respect to these parts, has no meaning. A non-distributive operator is in no sense a quaternion, since it does not obey the laws of quaternions (which are associative and distributive), and since no operational meaning can here be attached to the quadrinomial form so important in quaternions.

The reader who studies out the problems of the remaining 20 pages will do well to avoid an increasing sense of irritation. Very little use is made of the principles first laid down; but a great deal of a mysterious looking but trivial notation for loci, in no way a necessary part of the system. The problem work has no advantage in compactness of reasoning over the usual analytic geometry. In this respect it endures no comparison with the pages of Hamilton or Grassmann, of Heaviside or Gibbs. No doubt it is too much to expect such a test of a brief monograph, but one naturally assumes the problems to have been chosen so as to show the method at its best.

FRANK L. HITCHCOCK.

Mechanics of Particles and Rigid Bodies. By J. PRESCOTT. London, Longmans, Green, and Company, 1913. viii+535 pp.

THIS book has been designed to meet the needs of students aiming for a pass degree at a British university and contains all that they require in the subject of applied mechanics except hydrostatics.

Practically all English texts on mechanics include long lists of problems; many of them consist principally of illustrative examples and problems. This volume, however, presents a systematic development of the theory in which no pains have been spared to make the proofs rigorous enough for pure science, while the practical side of the subject has not been neglected. The problems following each chapter have evidently been chosen with great care and cover a wide range. Some of them demand merely substitution in formulas and numerical calculation, while others offer considerable theoretical difficulty. A special feature is that the answer is given to nearly every question.

An elementary course in the calculus is presupposed for the study of this text and the author appreciates the fact that the student who is applying the calculus for the first time to

physical problems has great difficulty in turning his formulas into numbers. Accordingly the examples worked in the text include several numerical ones, and some attention is given to putting results of theory in a form convenient for calculation. The book has a utilitarian bias which should make it useful to the engineering student, as well as interesting and live to the student of pure science. This attitude is emphasized throughout and is especially noticeable in the chapter on motion in two dimensions, where practical numerical applications are made of almost every topic treated, and in the applications to problems of astronomy.

The book begins with the subject of statics and, in addition to usual topics under this head, Part I contains chapters on graphical methods, moments of inertia, potential and attraction, and an extensive chapter on elasticity including strength of materials. Part II is devoted to the dynamics of a particle and the usual special types of motion are treated in excellent style. Besides a fairly complete chapter on central forces there is one on the problem of two or more bodies. The dynamics of a rigid body forms Part III and this is concerned principally with motion parallel to a fixed plane. There is also a chapter on motion in three dimensions of a rigid body with a symmetrical axis. The last chapter is on units and dimensions. If the student is to use the idea of dimensions as a check on his equations, it seems better to place this subject in the earlier part of the book.

W. R. LONGLEY.

Annuaire pour l'An 1915. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1914.

THE Annuaire bears no signs of the struggle which is taking place in Europe. It arrived about the usual time and contains the usual information brought up to date. The astronomical part is increased by a descriptive note, with maps, of the constellations and with the names and positions of the stars which form these groups. The editors frequently find it necessary to add new information and, in order to keep the volume within reasonable limits, to omit less needed tables and articles, publishing the latter only occasionally. This year the geographical positions of the principal towns on the earth are omitted as well as the long table of magnetic elements in France.

The single Notice is a long one by M. G. Bigourdan, "Les méthodes d'examen des miroirs et des objectifs." As is often the case in these appendices, both the older and the more modern methods are described, their mathematical bases are explained and applications to various instruments are made, so that the article practically constitutes a treatise on the subject in a form which can be read and understood without serious difficulty.

E. W. BROWN.

NOTES.

AT the meeting of the London mathematical society on March 11 the following papers were read: By E. W. HOBSON: "Some theorems in the theory of series of orthogonal functions"; by P. A. MACMAHON: "Investigations in the theory of the partition of numbers by a new method of partial fractions"; by R. L. HIPPISLEY: "Reciprocal and parallelogram linkages"; by J. R. WILSON: "A pseudo-sphere the equation of which is expressible in terms of elliptic functions"; by T. C. LEWIS: "Circles, spheres, etc., associated with a triangle, orthocentric tetrahedron, etc."

THE fifth ordinary meeting of the Edinburgh mathematical society for the session was held at the University of Glasgow on March 12. The following papers were read: By A. W. H. THOMPSON: "Solid geometry"; by G. D. C. STOKES: "A simple link apparatus for the mechanical solution of quadratic equations"; by F. TAVANI: "New formulae about the theory of the series of alternate sign."

BEGINNING with the April number, the staff of associate editors of the *Transactions of the American Mathematical Society* will include Professors A. B. COBLE and W. A. HURWITZ in place of Professors J. I. HUTCHINSON and MAX MASON, who have served since 1902 and 1911 respectively.

THE March number of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "A note on functions of lines," by G. A. BLISS; "A classification of quadratic vector functions," by F. L. HITCHCOCK. The April number of the *Proceedings* contains:

"Groups possessing at least one set of independent generators composed of as many operators as there are prime factors in the order of the group," by G. A. MILLER; "The second derivatives of the extremal integral for a general class of problems of the calculus of variations," by ARNOLD DRESDEN; "Point sets and allied Cremona groups," by A. B. COBLE; "The straight lines on modular cubic surfaces," by L. E. DICKSON.

THE Theory of Functions of a Complex Variable, by E. J. TOWNSEND, is announced by Henry Holt and Company as in press and to appear in June.

THE Smith prizes of Cambridge University for 1915 have been awarded as follows: to H. GLAUERT, of Trinity College, for his essay "On the elliptical form of a rotating fluid as disturbed by a satellite," and to H. JEFFREYS, of St. John's College, for his essays "Certain hypotheses as to the internal structure of the earth and moon" and "On a possible distribution of meteors." The Rayleigh prize was awarded to J. PROUDMAN, of Trinity College, for his papers on tidal motion.

THE Adams prize commission announces as the subject of the prize competition for 1915–1916 the following:

"The course of evolution of the configurations possible for a rotating and gravitating mass, including the discussion of the stabilities of the various forms."

THE first award of the Ackermann-Teubner memorial prize in mathematics has been made, in accordance with the wish of the founder, to Professor FELIX KLEIN.

THE royal Venetian institute announces the following prize problem for 1917:

To make some notable extension to the theory of periodic solutions of differential systems. It is not always possible to profit by the classical method of Poincaré, that of varying the parameters in a known solution, or at least, only for small variations. When this method does not apply, the only concrete result concerning the condition of existence is the criterion of Whittaker. The most desirable investigation would be the elucidation of the law of distribution of the periodic solutions within the field of the general integral.

The competition is unrestricted. Memoirs should be written in Italian, French, English, or German, and submitted to the institute before December 31, 1917. The value of the prize is 3000 lire.

UNIVERSITY OF STRASSBURG.—The following courses in mathematics are being given during the present summer semester:—By Professor F. SCHUR: Differential geometry, three hours; Selected portions of projective geometry, two hours; Seminar, two hours.—By Professor G. FABER: Theory of functions of a complex variable, three hours; Partial differential equations, two hours; Seminar, two hours.—By Professor M. SIMON: Non-euclidean geometry, two hours.—By Professor J. WELLSTEIN: Differential and integral calculus, three hours; Mechanical integration, two hours.—By Professor L. v. MISES: Descriptive geometry, with exercises, three hours.—By Professor P. EPSTEIN: Analytic geometry of space, three hours; Introduction to determinants, two hours.—By Dr. A. SPEISER: Theory of numbers, two hours.—By Professor J. BAUSCHINGER: New methods in celestial mechanics, three hours.

UNIVERSITY OF CHICAGO.—The following courses are announced for the summer quarter of 1915.—All courses are five hours a week.—By Professor E. H. MOORE: Synthetic projective geometry (first term); General analysis (first term).—By Professor A. C. LUNN: Theory of functions; Theory of relativity.—By Professor G. A. BLISS: Definite integrals; Calculus of variations.—By Professor W. D. MACMILLAN: Differential equations; Celestial mechanics.—By Professor A. DRESDEN: Analytic projective geometry (second term); Determinants and solid analytic geometry.—By Professor J. W. A. YOUNG: Differential calculus; Theory of numbers.—By Professor R. D. CARMICHAEL: Integral calculus; Differential equations (second term).—By Professor E. J. WILCZYNSKI: Projective differential geometry.—By Professor F. R. MOULTON: Theory of rotating bodies.

COLUMBIA UNIVERSITY.—Summer session, July 6 to August 13.—By Professor JAMES MACLAY: Higher algebra; Functions of a complex variable.—By Professor EDWARD KASNER: Theory of geometric constructions; Differential geometry.—By Professor W. B. FITE: Projective geometry; Differential equations.

THE following advanced courses in mathematics are announced for the academic year 1915–1916.

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Differential equations, four hours, second half-year.—By Professor F. N. COLE: Theory of groups, three hours, second half-year.—By Professor C. J. KEYSER: Modern theories in geometry, four hours; Mathematics, three hours.—By Professor D. E. SMITH: History of mathematics, four hours.—By Professor JAMES MACLAY: Elliptic functions, four hours, first half-year; Applications of elliptic functions, two hours, second half-year.—By Professor EDWARD KASNER: Differential equations, four hours, first half-year; Seminar in differential geometry, two hours.—By Professor W. B. FITE: Infinite series, three hours, first half-year.—By Professor H. E. HAWKES: Differential geometry of curves, three hours, first half-year.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of probabilities, three hours.—By Professor J. H. TANNER: Algebraic equations, three hours.—By Professor V. SNYDER: Geometry on an algebraic surface, three hours.—By Professor F. R. SHARPE: Vector analysis, three hours.—By Professor D. C. GILLESPIE: Projective geometry, three hours.—By Dr. C. F. CRAIG: Advanced analysis, three hours.—By Dr. F. W. OWENS: Differential equations, with applications, three hours.—By Dr. J. V. MCKELVEY: Mathematical pedagogy, three hours.—By Professor W. A. HURWITZ: Integral equations, three hours.—By Dr. L. L. SILVERMAN: Coordinate geometry, three hours.—By Dr. J. SLEPIAN: Mechanics, three hours.

HARVARD UNIVERSITY.—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Introduction to potential functions and Laplace's equation*; Elastic vibrations, Fourier's series, and Bessel's functions*; Theory of functions of several complex variables.—By Professor M. BÔCHER: Modern geometry and modern algebra; Theory of functions.—By Professor C. L. BOUTON: Differential equations and Lie's theory of continuous groups.—By Professor J. L. COOLIDGE: Subject-matter of elementary mathematics*; Probability*; Projective geometry*; Non-Euclidean geometry*.—By Professor H. N. DAVIS: Second course in dy-

namics.—By Professor G. D. BIRKHOFF: Infinite series and products*; Problem of three bodies.—By Dr. D. JACKSON: Advanced calculus; Theory of numbers*.—By Dr. G. M. GREEN: Differential geometry of curves and surfaces*; Projective differential geometry*.—By Dr. E. KIRCHER: Properties of polynomials, and invariants*; Algebraic numbers*.—By Dr. G. A. PFEIFFER: Linear partial differential equations of the second order*; Conformal transformations*.

Professor BIRKHOFF and Dr. JACKSON will conduct a fortnightly seminar in analysis.

Courses of research are also offered by Professor OSGOOD in the theory of functions, by Professor BÔCHER in analysis and algebra, by Professor BOUTON in the theory of point transformations, by Professor COOLIDGE in geometry, by Professor BIRKHOFF in differential equations, by Dr. JACKSON in the theory of functions of a real variable, and by Dr. GREEN in differential geometry.

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry, three hours (first half year); Vector analysis, three hours (second half year); Seminar, one hour.—By Professor A. B. COBLE: Algebraic functions, two hours.—By Professor A. COHEN: Advanced differential equations, two hours; Theory of functions, two hours.

PRINCETON UNIVERSITY.—By Professor H. B. FINE: Algebraic functions, three hours.—By Professor L. P. EISENHART: Mechanics, three hours (first term); Projective geometry, three hours (second term).—By Professor O. VEBLEN: Theory of sets of points, three hours; Coordinate geometry, three hours; Advanced analysis, three hours.—By Professor E. P. ADAMS: Electricity and magnetism, three hours; Analytic mechanics, three hours.—By Professor T. H. GRONWALL: Conformal representation, three hours; Differential equations and advanced calculus, three hours.—By Mr. J. W. ALEXANDER: Birational transformations, three hours (first term).—By Mr. A. A. BENNETT: Elliptic functions, three hours (second term).—By Dr. H. GALAJIKIAN: Differential equations of mathematical physics, three hours (first term).

YALE UNIVERSITY.—By Professor J. PIERPONT: Theory of functions of a complex variable, three hours; Modern analytic

geometry, three hours; Modern higher geometry, three hours.—By Professor P. F. SMITH: Continuous groups of transformations, three hours.—By Professor E. W. BROWN: Advanced calculus, three hours; Statics and dynamics, three hours; Advanced dynamics, three hours.—By Professor W. R. LONGLEY: Potential theory and harmonic analysis, three hours; Integral equations, three hours.—By Professor W. A. WILSON: Theory of functions of real variables, three hours.—By Dr. D. D. LEIB: Advanced algebra, three hours.—By Dr. G. M. CONWELL and Dr. H. F. MACNEISH: Differential geometry, three hours.—By Dr. E. J. MILES: Calculus of variations, three hours; Advanced calculus of variations, three hours.

THE *Jahresbericht der Deutschen Mathematiker-Vereinigung* notes the following extraordinary series of anniversaries of German professors of mathematics: M. NOETHER, of Erlangen, and A. WANGERIN, of Halle, celebrated their seventieth birthdays on September 24, and November 18, 1914, respectively. The eightieth birthday of E. SELLING, of Munich, fell on November 5, and the eighty-fifth birthday of M. CANTOR, of Heidelberg, on August 23. H. A. SCHWARZ, of Berlin, and J. THOMAE, of Jena, passed the fiftieth anniversary of their doctorates on August 6, and Professor E. LAMPE, of Berlin, on December 21.

PROFESSOR H. S. WHITE, of Vassar College, has been elected a member of the National academy of sciences.

PROFESSORS J. A. MILLER, of Swarthmore College, and W. F. OSGOOD, of Harvard University, have been elected members of the American philosophical society.

PROFESSOR T. S. FISKE has been appointed administrative head of the department of mathematics at Columbia University, succeeding Professor C. J. KEYSER, who has been relieved of administrative duties at his own request.

AT the University of Oklahoma Professor F. C. KENT has resigned and Dr. H. C. GOSSARD has been appointed instructor in mathematics.

At the University of Michigan Professor L. C. KARPINSKI has been promoted to a junior professorship of mathematics.

At the Massachusetts institute of technology, Dr. H. B. PHILLIPS has been promoted to an assistant professorship of mathematics.

MR. C. H. YEATON has been appointed instructor in mathematics at Dartmouth College.

At Princeton University Dr. H. GALAJIKIAN has been appointed instructor in mathematics.—Mr. H. L. BARUCH has been appointed assistant in mathematics.

THE many friends of Professors KLEIN and RUNGE, of Göttingen, will be deeply grieved to learn of the death on the battlefield of the former's youngest son-in-law and the latter's younger son.

PROFESSOR F. A. SHERMAN, who held the chair of mathematics at Dartmouth College from 1871 to 1911, died February 25 at the age of seventy-four years.

PROFESSOR A. E. HAYNES, who retired from his professorship of mathematics at the University of Minnesota in 1911 after eighteen years service, died on March 12 at the age of sixty-six years.

BOOK catalogues: W. Heffer and Sons, Cambridge, England, catalogue 132, including about 800 titles in mathematics, physics, and engineering.—Galloway and Porter, Cambridge, England, catalogue 76, short list of mathematical books, 75 titles.—R. W. Lull, 84 Hanover Street, Manchester, N. H., list of 68 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMALDI (U.). Sulla evoluzione della matematica come strumento d'indagine: discorso per l'inaugurazione dell'anno accademico nella r. Università di Modena il 5 novembre, 1914. Modena, Soc. tip. Modenese, 1914. 8vo. 35 pp.

BIELANIN (F. A. P.). See FILIPPOWITSCH (F. W.).

BOLTI (L.). L'infinito. Genova, Formiggini, 1912. 8vo. 5+531 pp.

CARR (H. W.). Henri Bergson: The philosophy of change. London and Edinburgh, Jack; New York, The Dodge Publishing Co., 1912. 92 pp.

CATALOGUE of scientific papers (1884–1900). Compiled by the Royal Society of London. Vol. 14, C-Fittig. Cambridge, University Press, 1915. 4to. 1024 pp. 2£ 10s.

CIANI (E.). Il metodo delle coordinate proiettive omogenee nello studio degli enti algebrici (seguito alle Lezioni di geometria proiettiva ed analitica). Pisa, Spoerri (F. Mariotti), 1915. 8vo. 215 pp.+2 tables. L. 10.00

COFFEY (P.). The science of logic. An inquiry into the principles of accurate thought and scientific method. London, Longmans, 1912. 20+445+7+359 pp.

COUTURAT (R.). Philosophical principles of mathematics. Translated from the French into Russian by B. Koren, under the direction of P. S. Juschkewitsch. St. Petersburg, 1912. 8vo. 260 pp.

FILIPPOWITSCH (F. W.). Guide through mathematical literature. Compiled by F. W. F. with the assistance of F. A. P. Bielanin and S. G. Schiperko. (In Russian.) St. Petersburg, 1912. 8vo. 50 pp.

GUASTELLA (C.). L'infinito. From Annuario della biblioteca filosofica. Palermo, Reber, 1912. 170 pp.

HERBERTZ (R.). Die Philosophie des Raumes. Oeffentl. Hochschul-Vortrag. Stuttgart, W. Spemann, 1912. Gr. 8vo. 38 pp.

HILTON (H.). Homogeneous linear substitutions. Oxford, Clarendon Press, 1915. 8+184 pp. 12s. 6d.

JUSCHKEWITSCH (P. S.). See COUTURAT (R.).

KOREN (B.). See COUTURAT (R.).

LOEWY (A.). Lehrbuch der Algebra. 1ter Teil: Grundlagen der Arithmetik. Leipzig, Veit, 1915. 398 pp. Geh. M. 12.00

SCHIPERKO (S. G.). See FILIPPOWITSCH (F. W.).

SINTZOV (D.). Russian mathematical bibliography. Part 2 for the year 1909. Edited by D. Sintzov. (In Russian.) Odessa, "Mathesis," 1912. 8vo. 15+28+10 pp.

STIELTJES (T. J.). Œuvres complètes de Thomas Jan Stieltjes publiées par les soins de la Société mathématique d'Amsterdam. Tome 1. Groningen, P. Noordhoff, 1914. 4to. 472 pp.

WESTAWAY (F. W.). Scientific method: its philosophy and its practice. London, Blackie, 1912. 21+439 pp.

WILDBRETT (A.). Algebraische Analysis, Algebra und Infinitesimalrechnung. 2ter Teil: Infinitesimalrechnung. Nürnberg, Korn, 1914. 199 pp. Geb. M. 3.80

ZAREMBA (S.). Arytmetyka teoretyczna. Krakau, Verlag der K. K. Akademie der Wissenschaften, 1912. 8vo. 20+859 pp.

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). A shilling arithmetic. London, Bell, 1915. 8vo. 1s.

BARRACLOUGH (A.). See CRACKNELL (A. G.).

BÖRNER (—.). See SUPPANTSCHITSCH (R.).

BOURNE (A. A.). See BAKER (W. M.).

CRACKNELL (A. G.) and BARRACLOUGH (A.). Junior algebra. London, University Tutorial Press, 1915. 6+280 pp. 2s. 6d.

GOTHE (G.). Rechenbuch für Stadtschulen. Leipzig, G. Freytag, 1912. 8vo. Unterstufe, 1. Heft, 46 pp. Ausgabe für Knabenschulen, I. Teil, 2. Heft, 127 pp. Ausgabe für Knabenvolksschulen, II. Teil, Mittelstufe, 216 pp. III. Teil, Oberstufe, 184 pp. Ausgabe für Knabenmittelschulen, II. Teil, 242 pp. III. Teil, Oberstufe, 206 pp. Ausgabe für Mädchenschulen, I. Teil, Unterstufe, 2. Heft, 127 pp.; II. Teil, Mittelstufe, 215 pp. Ausgabe für Mädchenvolksschulen, III. Teil, Oberstufe, 192 pp. Ausgabe für Mädchenmittelschulen, III. Teil, Oberstufe, 214 pp.

HOFFMANN (T.). Ueber fehlerhafte mathematische Ausdrucksweise. Progr. 798, Realgymn. Zwickau, 1912. 29 pp.

IGNATJEV (E. M.). Im Reiche des Scharfsinnes oder Arithmetik für alle. Ein Buch für Haus und Schule. Versuch einer mathematischen Chrestomathie. Band 2. 2te durchges. u. vermehrte Auflage. (Russisch.) St. Petersburg, 1912. 8vo. 290 pp.

JASCHUNSKI (J.). See SIMON (M.).

KNOCHENDÖPPEL (C.). See MÖHLE (F.).

LAISANT (C. A.). Introduction to mathematics. Translated from the French into Russian under the direction of W. W. Mropzek. St. Petersburg, 1912. 16mo. 185+2 pp.

LIETZMANN (W.). See WEINREICH (H.).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, September–November, 1914. Edited by R. M. Milne. London, Macmillan, 1915. 8vo. 28 pp. 1s.

MÖHLE (F.) und KNOCHENDÖPPEL (C.). Lehr- und Uebungsbuch der Mathematik für Lyzeen und höhere Mädchenschulen. 1ter Teil: für Klasse IV und III. Breslau, Hirt, 1915. 176 pp. Geb. M. 2.25

MROPZEK (W. W.). See LAISANT (C. A.).

ROHRBERG (A.). Der mathematische Unterricht in Dänemark. (Berichte und Mitteilungen der Intern. Math. Unterrichts-Kommission, 2te Folge, Nr. 1.) Leipzig, Teubner, 1915. 8vo. 6+54 pp. M. 2.40

SIMON (M.). Didaktik und Methodik des Rechnens und der Mathematik. Uebersetzt von J. Jaschunski. (Russisch.) St. Petersburg, "Physice," 1912. 8vo. 4+257 pp.

SSAMOCHWALOV (P.). Zum Unterricht der Analysis in Kadettenkorps. St. Petersburg, 1912. 8vo. 28 pp.+8 tables.

SUPPANTSCHITSCH (R.) und BÖRNER (—.). Mathematisches Unterrichtswerk, für Mädchenschulen bearbeitet. Wien, Tempsky, 1914. Arithmetik für Mädchenlyzeen, Heft 4, 106 pp. Heft 5, 63 pp. Geb. Kr. 1.80 + 1.10.

WEINREICH (H.). Die Fortschritte der mathematischen Unterrichtsreform seit 1910, sowie Lietzmann (W.), Der Pariser Kongress der International Math. Unterrichts-Kommission vom 1.–4. April, 1914. Leipzig, Teubner, 1915. Pp. 219–310. M. 3.00

WOLFF (G.). Der mathematische Unterricht in England. (Berichte und Mitteilungen der Intern. Math. Unterrichts-Kommission, 2te Folge, Nr. 2). Leipzig, Teubner, 1915. 8vo. 6 + 205 pp.

III. APPLIED MATHEMATICS.

AUERBACH (F.). See WARBURG (E.).

BARTLETT (D. P.). General principles of the method of least squares, with applications. 3d edition. Boston, D. P. Bartlett, 1915. 8vo. 142+12 pp. Cloth. $2.25

BRAUN (F.). See WARBURG (E.).

COAN (C. A.). See COLMAN (S.).

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